Technische Universität München **Winter term 2017/18**
I7
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Automata and Formal Languages — Homework 1

Due 24.10.2017

Download `JFLAP` from www.jflap.org. We will use the *finite automata* and *regular expression* modes.

Exercise 1.1
Let $L = \{w \in \{a, b, c\}^* : w \text{ starts with } ac \text{ and ends with } ab\}$.

(a) Give an NFA that accepts L.

(b) Give a DFA that accepts L.

(c) Give a regular expression for L.

(d) Use `JFLAP` to convert your NFA of (a) and your regular expression of (c) to DFAs.

Exercise 1.2
Let $\text{msbf}\colon \{0,1\}^* \to \mathbb{N}$ and $\text{lsbf}\colon \{0,1\}^* \to \mathbb{N}$ be such that $\text{msbf}(w)$ and $\text{lsbf}(w)$ are respectively the number represented by w in the "most significant bit first" and "least significant bit first" encoding. For example,

$$\begin{aligned} \text{msbf}(1010) &= 10, & \text{msbf}(100) &= 4, & \text{msbf}(0011) &= 3,\\ \text{lsbf}(1010) &= 5, & \text{lsbf}(100) &= 1, & \text{lsbf}(0011) &= 12. \end{aligned}$$

For every $n \geq 2$, let us define the following languages:

$$\begin{aligned} M_n &= \{w \in \{0,1\}^* : \text{msbf}(w) \text{ is a multiple of } n\},\\ L_n &= \{w \in \{0,1\}^* : \text{lsbf}(w) \text{ is a multiple of } n\}. \end{aligned}$$

(a) Give DFAs and regular expressions for M_2, L_2 and $M_2 \cap L_2$.

(b) Give DFAs and regular expressions for M_4, L_4 and $M_4 \cap L_4$.

(c) Give a DFA that accepts M_3. [Hint:]

(d) Give a DFA that accepts L_3. [Hint:]

(e) What can you say about $M_3 \cap L_3$?

(f) Use `JFLAP` to obtain a regular expression for M_3.

(g) Give a general DFA construction for M_n where $n \geq 2$.

Exercise 1.3
The *reverse* of a word $w \in \Sigma^*$ is defined as

$$w^R = \begin{cases} \varepsilon & \text{if } w = \varepsilon,\\ a_n a_{n-1} \cdots a_1 & \text{if } w = a_1 a_2 \cdots a_n \text{ where each } a_i \in \Sigma. \end{cases}$$

The *reverse* of a language $L \subseteq \Sigma^*$ is defined as $L^R = \{w^R : w \in L\}$.

(a) Let A be an NFA. Describe an NFA B such that $L(B) = L(A)^R$.

(b) Does your construction in (a) works for DFAs as well? More precisely, does it preserve determinism?

(c) Show that $M_n = (L_n)^R$ for every $n \geq 2$.

Exercise 1.4
Let A and B be DFAs over some alphabet Σ.

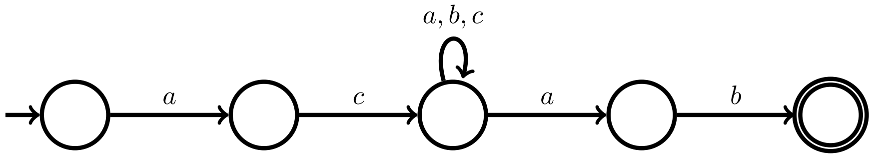
(a) Describe DFAs C and D such that $L(C) = L(A) \cup L(B)$ and $L(D) = L(A) \cap L(B)$.

(b) Prove that D is correct, i.e. that indeed $L(D) = L(A) \cap L(B)$.

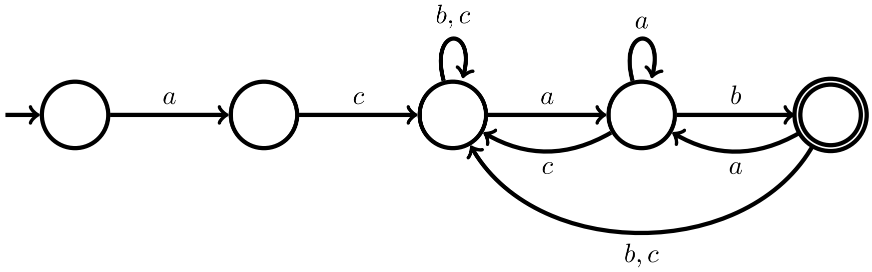
(c) If A and B were NFAs, could you construct NFAs with fewer states for union and intersection? Explain your answer.

Solution 1.1

(a)



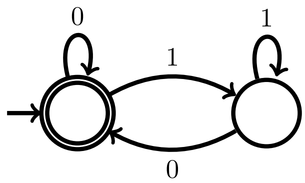
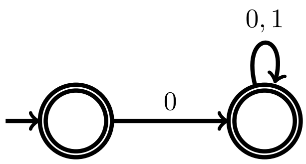
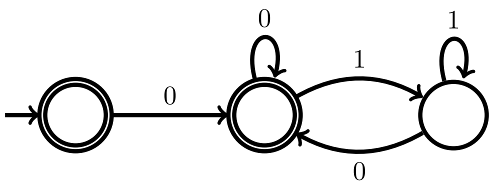
(b)



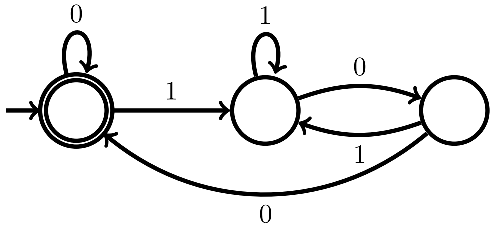
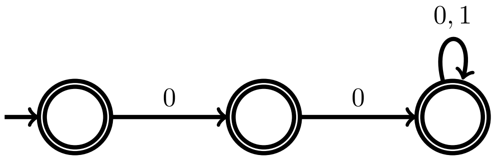
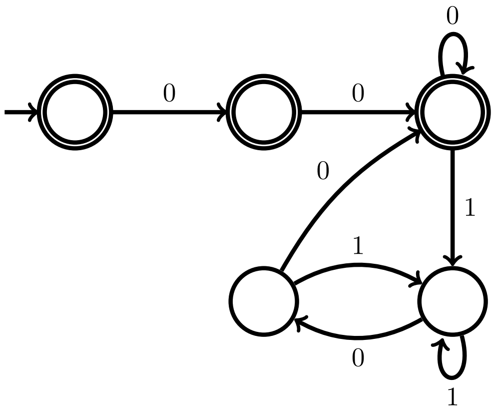
(c) $ac(a+b+c)^*ab$

Solution 1.2

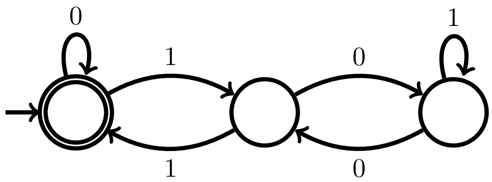
(a)

	DFA	Regular expression
M_2:		$\varepsilon + (0+1)^*0$
L_2:		$\varepsilon + 0(0+1)^*$
$M_2 \cap L_2$:		$\varepsilon + 0 + 0(0+1)^*0$

(b)

	DFA	Regular expression
M_4:		$\varepsilon + 0 + (0+1)^*00$
L_4:		$\varepsilon + 0 + 00(0+1)^*$
$M_4 \cap L_4$:		$\varepsilon + 0 + 0(0+1)^*0$

(c)



(d) The automaton for L_3 is the same as the one given in (c).

(e) $M_3 \cap L_3 = M_3 = L_3$.

(f) `JFLAP` yields $(0 + 11 + 10(1 + 00)^*01)^*$.

(g) The automaton for M_n is defined as $A_n = (Q_n, \{0,1\}, \delta_n, 0, \{0\})$ where

$$Q_n = \{0, 1, \ldots, n-1\},$$
$$\delta_n(q, b) = (2q + b) \bmod n \quad \text{for every } q \in Q_n \text{ and } b \in \{0,1\}.$$

Solution 1.3

(a) We reverse the transitions of A and swap its initial and final states. More formally, let $A = (Q, \Sigma, \delta, Q_0, F)$. We define B as $B = (Q, \Sigma, \delta', F, Q_0)$ where $\delta'(p, a) = \{q \in Q : p \in \delta(q, a)\}$.

(b) No, if A is deterministic, then B is not necessarily deterministic. For example, the construction applied to the DFA of #1.2(a) for M_2 does not yield a DFA.

(c) We have

$$\begin{aligned}
w \in M_n &\iff \text{msbf}(w) \equiv 0 \pmod{n} && \text{(by def. of } M_n) \\
&\iff \text{lsbf}(w^R) \equiv 0 \pmod{n} && (\text{by msbf}(w) = \text{lsbf}(w^R)) \\
&\iff w^R \in L_n && \text{(by def. of } L_n) \\
&\iff w \in L_n^R && (\text{by } u \in L_n \iff u^R \in (L_n)^R \text{ and } (u^R)^R = u).
\end{aligned}$$

Solution 1.4

(a) Let $A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_0', F_B)$. We define C and D as follows:

$$C = (Q_A \times Q_B, \Sigma, \delta', (q_0, q_0'), F_C),$$
$$D = (Q_A \times Q_B, \Sigma, \delta', (q_0, q_0'), F_D),$$

where $\delta'((p,q),a) = (\delta_A(p,a), \delta_B(q,a))$ and

$$F_C = \{(p,q) \in Q_A \times Q_B : p \in F_A \vee q \in F_B\},$$
$$F_D = \{(p,q) \in Q_A \times Q_B : p \in F_A \wedge q \in F_B\}.$$

(b) It suffices to prove that

$$(p,q) \xrightarrow{w}_D (p',q') \iff p \xrightarrow{w}_A p' \text{ and } q \xrightarrow{w}_B q'.$$

We proceed by induction on $|w|$. If $|w| = 0$, then $w = \varepsilon$ and the claim trivially holds. Assume that $|w| > 0$ and suppose that the claim holds for every word of length $|w| - 1$. There exist $a \in \Sigma$ and $u \in \Sigma^*$ such that $w = au$. We have,

$$\begin{aligned}
(p,q) \xrightarrow{w}_D (p',q') &\iff \delta'((p,q),a) = (p'',q'') && \text{and } (p'',q'') \xrightarrow{u}_D (p',q') \\
&\iff \delta'((p,q),a) = (p'',q'') && \text{and } p'' \xrightarrow{u}_A p' \text{ and } q'' \xrightarrow{u}_B q' \quad \text{(by ind. hyp.)} \\
&\iff \delta_A(p,a) = p'' \text{ and } \delta_B(q,a) = q'' && \text{and } p'' \xrightarrow{u}_A p' \text{ and } q'' \xrightarrow{u}_B q' \quad \text{(by def. of } C) \\
&\iff p \xrightarrow{au}_A p' \text{ and } q \xrightarrow{au}_B q' \\
&\iff p \xrightarrow{w}_A p' \text{ and } q \xrightarrow{w}_B q'.
\end{aligned}$$

□

(c) Intersection sometimes require the $|Q_A| \cdot |Q_B|$ states from the product construction (e.g. see [1, Thm. 11]), however it is possible to do better with union. Since multiple initial states are allowed in this course, we can build the following NFA:

$$C = (Q_A \cup Q_B, \Sigma, \delta_A \cup \delta_B, Q_0 \cup Q_0', F_A \cup F_B).$$

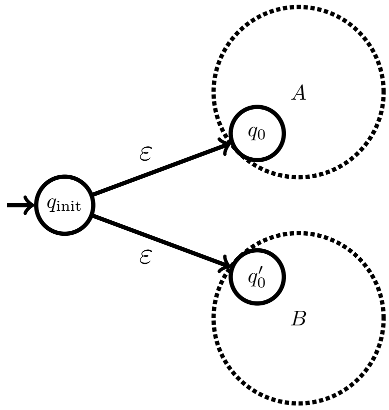
If we were restricted to a single initial state, we could build the following NFA:

$$C = (q_{\text{init}} \cup Q_A \cup Q_B, \Sigma, \delta', q_{\text{init}}, F_A \cup F_B)$$

where

$$\delta'(q,a) = \begin{cases} \delta_A(q_0,a) \cup \delta_B(q_0',a) & \text{if } q = q_{\text{init}}, \\ \delta_A(q,a) & \text{if } q \in Q_A, \\ \delta_B(q,a) & \text{if } q \in Q_B. \end{cases}$$

The last construction would even be simpler if ε-transitions were allowed:



References

[1] Markus Holzer and Martin Kutrib. State complexity of basic operations on nondeterministic finite automata. In *Proc. 7^{th} International Conference on Implementation and Application of Automata (CIAA)*, pages 148–157, 2002.