Technische Universität München **Winter term 2016/17**
I7
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Automata and Formal Languages — Homework 13

Due 03.02.2017

Exercise 13.1
Show that

(a) $\neg\mathbf{X}\varphi \equiv \mathbf{X}\neg\varphi$

(b) $\neg\mathbf{F}\varphi \equiv \mathbf{G}\neg\varphi$

(c) $\neg\mathbf{G}\varphi \equiv \mathbf{F}\neg\varphi$

(d) $\mathbf{XF}\varphi \equiv \mathbf{FX}\varphi$

(e) $\mathbf{XG}\varphi \equiv \mathbf{GX}\varphi$

Exercise 13.2
Let $\mathrm{AP} = \{p, q, r\}$. Give formulas that hold for the computations satisfying the following properties:

(a) p is false before q

(b) p becomes true before q

(c) p is true between q and r

(d) only p is true at even positions and only q is true at odd positions.

Exercise 13.3
Prove or disprove the following distributivity properties:

(a) $\mathbf{X}(\varphi \vee \psi) \equiv \mathbf{X}\varphi \vee \mathbf{X}\psi$

(b) $\mathbf{X}(\varphi \wedge \psi) \equiv \mathbf{X}\varphi \wedge \mathbf{X}\psi$

(c) $\mathbf{X}(\varphi \;\mathbf{U}\; \psi) \equiv (\mathbf{X}\varphi) \;\mathbf{U}\; (\mathbf{X}\psi)$

(d) $\mathbf{F}(\varphi \vee \psi) \equiv \mathbf{F}\varphi \vee \mathbf{F}\psi$

(e) $\mathbf{F}(\varphi \wedge \psi) \equiv \mathbf{F}\varphi \wedge \mathbf{F}\psi$

(f) $\mathbf{G}(\varphi \vee \psi) \equiv \mathbf{G}\varphi \vee \mathbf{G}\psi$

(g) $\mathbf{G}(\varphi \wedge \psi) \equiv \mathbf{G}\varphi \wedge \mathbf{G}\psi$

(h) $\mathbf{GF}(\varphi \vee \psi) \equiv \mathbf{GF}\varphi \vee \mathbf{GF}\psi$

(i) $\mathbf{GF}(\varphi \wedge \psi) \equiv \mathbf{GF}\varphi \wedge \mathbf{GF}\psi$

(j) $\rho \;\mathbf{U}\; (\varphi \vee \psi) \equiv (\rho \;\mathbf{U}\; \varphi) \vee (\rho \;\mathbf{U}\; \psi)$

(k) $(\varphi \vee \psi) \;\mathbf{U}\; \rho \equiv (\varphi \;\mathbf{U}\; \rho) \vee (\psi \;\mathbf{U}\; \rho)$

(l) $\rho \;\mathbf{U}\; (\varphi \wedge \psi) \equiv (\rho \;\mathbf{U}\; \varphi) \wedge (\rho \;\mathbf{U}\; \psi)$

(m) $(\varphi \wedge \psi) \;\mathbf{U}\; \rho \equiv (\varphi \;\mathbf{U}\; \rho) \wedge (\psi \;\mathbf{U}\; \rho)$

Exercise 13.4
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p \;\mathbf{U}\; (\neg p \vee q))$

(f) $\neg(p \;\mathbf{U}\; q) \leftrightarrow (\neg p \;\mathbf{U}\; \neg q)$

(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$

Exercise 13.5
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give LTL formulas for the following ω-languages:

(a) $\{p, q\}\, \emptyset\, \Sigma^\omega$

(b) $\Sigma^*\, \{q\}^\omega$

(c) $\Sigma^*\, (\{p\} + \{p, q\})\, \Sigma^*\, \{q\}\, \Sigma^\omega$

(d) $\{p\}^*\, \{q\}^*\, \emptyset^\omega$

Exercise 13.6
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give Büchi automata for the ω-languages over Σ defined by the following LTL formulas:

(a) $\mathbf{XG}\neg p$

(b) $(\mathbf{GF}p) \rightarrow (\mathbf{F}q)$

(c) $p \wedge \neg(\mathbf{XF}p)$

(d) $\mathbf{G}(p\ \mathbf{U}\ (p \rightarrow q))$

(e) $\mathbf{F}q \rightarrow (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

Solution 13.1

(a)

$$\begin{aligned}\sigma \models \neg\mathbf{X}\varphi &\iff \sigma \not\models \mathbf{X}\varphi\\ &\iff \sigma^1 \not\models \varphi\\ &\iff \sigma^1 \models \neg\varphi\\ &\iff \sigma \models \mathbf{X}\neg\varphi.\end{aligned}$$

(b)

$$\begin{aligned}\sigma \models \neg\mathbf{F}\varphi &\iff \neg(\sigma \models \mathbf{F}\varphi)\\ &\iff \neg(\exists k \geq 0\ \sigma^k \models \varphi)\\ &\iff \forall k \geq 0\ \neg(\sigma^k \models \varphi)\\ &\iff \forall k \geq 0\ \sigma^k \models \neg\varphi\\ &\iff \mathbf{G}\neg\varphi.\end{aligned}$$

(c)

$$\begin{aligned}\sigma \models \neg\mathbf{G}\varphi &\iff \neg(\sigma \models \mathbf{G}\varphi)\\ &\iff \neg(\forall k \geq 0\ \sigma^k \models \varphi)\\ &\iff \exists k \geq 0\ \neg(\sigma^k \models \varphi)\\ &\iff \exists k \geq 0\ \sigma^k \models \neg\varphi\\ &\iff \mathbf{F}\neg\varphi.\end{aligned}$$

(d)

$$\begin{aligned}\sigma \models \mathbf{XF}\varphi &\iff \sigma^1 \models \mathbf{F}\varphi\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^1)^k \models \varphi\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k)^1 \models \varphi\\ &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models \mathbf{X}\varphi\\ &\iff \sigma \models \mathbf{FX}\varphi.\end{aligned}$$

(e)

$$\begin{aligned}\sigma \models \mathbf{XG}\varphi &\iff \sigma^1 \models \mathbf{G}\varphi\\ &\iff \forall k \geq 0\ (\sigma^1)^k \models \varphi\\ &\iff \forall k \geq 0\ (\sigma^k)^1 \models \varphi\\ &\iff \forall k \geq 0\ \sigma^k \models \mathbf{X}\varphi\\ &\iff \sigma \models \mathbf{GX}\varphi.\end{aligned}$$

Solution 13.2

(a) $\mathbf{F}q \rightarrow (\neg p\ \mathbf{U}\ q)$

(b) $\mathbf{F}q \rightarrow (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

(c) $\mathbf{G}((q \wedge \mathbf{F}r) \rightarrow \mathbf{X}(p\ \mathbf{U}\ r))$

(d) $\mathbf{G}(\neg r) \wedge \mathbf{G}(p \leftrightarrow \neg q) \wedge p \wedge G(p \rightarrow \mathbf{X}q) \wedge G(q \rightarrow \mathbf{X}p)$

Solution 13.3

(a–b) Both (a) and (b) hold. Let $\circ \in \{\vee, \wedge\}$. We have

$$\begin{aligned}\sigma \models \mathbf{X}(\varphi \circ \psi) &\iff \sigma^1 \models (\varphi \circ \psi)\\ &\iff (\sigma^1 \models \varphi) \circ (\sigma^1 \models \psi)\\ &\iff \sigma \models \mathbf{X}\varphi \ \circ \ \sigma \models \mathbf{X}\psi.\end{aligned}$$

(c) True, since:

$$\begin{aligned}\sigma \models \mathbf{X}(\varphi \mathbin{\mathbf{U}} \psi) &\iff \sigma^1 \models (\varphi \mathbin{\mathbf{U}} \psi)\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^1)^k \models \varphi \text{ and } \forall 0 \leq i < k \ (\sigma^1)^i \models \psi\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k)^1 \models \varphi \text{ and } \forall 0 \leq i < k \ (\sigma^i)^1 \models \psi\\ &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models \mathbf{X}\varphi \text{ and } \sigma^i \models \mathbf{X}\psi \text{ for every } 0 \leq i < k\\ &\iff \sigma \models (\mathbf{X}\varphi) \mathbin{\mathbf{U}} (\mathbf{X}\psi).\end{aligned}$$

(d) True, since:

$$\begin{aligned}\sigma \models \mathbf{F}(\varphi \vee \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \vee \psi)\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi) \vee (\sigma^k \models \psi)\\ &\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi) \vee (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi)\\ &\iff \sigma \models \mathbf{F}\varphi \vee \mathbf{F}\psi.\end{aligned}$$

(e) False. Let $\sigma = \{p\}\{q\}\emptyset^\omega$. We have $\sigma \models \mathbf{F}p \wedge \mathbf{F}q$ and $\sigma \not\models \mathbf{F}(\varphi \wedge \psi)$.

(f) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \models \mathbf{G}(p \vee q)$ and $\sigma \not\models \mathbf{G}p \vee \mathbf{G}q$.

(g) True, since:

$$\begin{aligned}\sigma \models \mathbf{G}(\varphi \wedge \psi) &\iff \forall k \geq 0 \ \sigma^k \models (\varphi \wedge \psi)\\ &\iff \forall k \geq 0 \ (\sigma^k \models \varphi) \wedge (\sigma^k \models \psi)\\ &\iff (\forall k \geq 0 \ \sigma^k \models \varphi) \wedge (\forall k \geq 0 \ \sigma^k \models \psi)\\ &\iff \sigma \models \mathbf{G}\varphi \wedge \mathbf{G}\psi.\end{aligned}$$

(h) True. If $\sigma \models \mathbf{GF}\varphi \vee \mathbf{GF}\psi$, then $\sigma \models \mathbf{GF}(\varphi \vee \psi)$. If $\sigma \models \mathbf{GF}(\varphi \vee \psi)$, then there exist $i_0 < i_1 < \cdots$ such that

$$\sigma^{i_j} \models \varphi \vee \psi \text{ for every } j \in \mathbb{N}. \tag{1}$$

Let $I = \{j \in \mathbb{N} : \sigma^{i_j} \models \varphi\}$ and $J = \{j \in \mathbb{N} : \sigma^{i_j} \models \psi\}$. If I and J are both finite, then (1) does not hold, which is a contradiction. Therefore, at least one of I and J is infinite. This implies that $\sigma \models \mathbf{GF}\varphi \vee \mathbf{GF}\psi$.

(i) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \not\models \mathbf{GF}(p \wedge q)$ and $\sigma \models \mathbf{GF}p \wedge \mathbf{GF}q$.

(j) True, since:

$$\begin{aligned}\sigma \models \rho \mathbin{\mathbf{U}} (\varphi \vee \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \vee \psi) \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho\\ &\iff \exists k \geq 0 \text{ s.t. } ((\sigma^k \models \varphi) \vee (\sigma^k \models \psi)) \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho) \vee (\sigma^k \models \psi \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho)\\ &\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho) \vee (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho)\\ &\iff \sigma \models (\rho \mathbin{\mathbf{U}} \varphi) \vee (\rho \mathbin{\mathbf{U}} \psi).\end{aligned}$$

(k) False. Let $\sigma = \{p\}\{q\}\{r\}\emptyset^\omega$. We have $\sigma \models (p \vee q) \mathbin{\mathbf{U}} r$ and $\sigma \not\models (p \mathbin{\mathbf{U}} r) \vee (q \mathbin{\mathbf{U}} r)$.

(l) False. Let $\sigma = \{r\}\{p,r\}\{q\}\emptyset^\omega$. We have $\sigma \not\models r \mathbf{U} (p \wedge q)$ and $\sigma \models (r \mathbf{U} p) \wedge (r \mathbf{U} q)$.

(m) True, since:

$$\begin{aligned}
\sigma \models (\varphi \wedge \psi) \mathbf{U} \rho &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models \rho \text{ and } \forall 0 \leq i < k\ \sigma^i \models (\varphi \wedge \psi) \\
&\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models \rho \text{ and } \forall 0 \leq i < k\ (\sigma^i \models \varphi \wedge \sigma^i \models \psi) \\
&\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \rho \text{ and } \forall 0 \leq i < k\ \sigma^i \models \varphi) \wedge (\sigma^k \models \rho \text{ and } \forall 0 \leq i < k\ \sigma^i \models \psi) \\
&\overset{(1)}{\iff} (\exists m \geq 0 \text{ s.t. } \sigma^m \models \rho \text{ and } \forall 0 \leq i < m\ \sigma^i \models \varphi) \wedge (\exists n \geq 0 \text{ s.t. } \sigma^n \models \rho \text{ and } \forall 0 \leq i < n\ \sigma^i \models \psi) \\
&\iff \sigma \models (\varphi \mathbf{U} \rho) \wedge (\psi \mathbf{U} \rho).
\end{aligned}$$

where $\overset{(1)}{\Longleftarrow}$ follows by taking $k = \min(m, n)$.

Solution 13.4

(a) $\mathbf{G}p \to \mathbf{F}q$ is a tautology since

$$\begin{aligned}
\sigma \models \mathbf{G}p &\iff \forall k \geq 0\ \sigma^k \models p \\
&\implies \exists k \geq 0\ \sigma^k \models q \\
&\iff \exists \sigma \models \mathbf{F}q.
\end{aligned}$$

(b) $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ is a tautology. For the sake of contradiction, suppose this is not the case. There exists σ such that

$$\sigma \models \mathbf{G}(p \to q), \text{ and} \tag{2}$$
$$\sigma \not\models (\mathbf{G}p \to \mathbf{G}q). \tag{3}$$

By (3), we have

$$\begin{aligned}
&\sigma \models \mathbf{G}p, \text{ and} \\
&\sigma \not\models \mathbf{G}q.
\end{aligned}$$

Therefore, there exists $k \geq 0$ such that $p \in \sigma(k)$ and $q \notin \sigma(k)$ which contradicts (2).

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$ is not a tautology since it is not satisfied by $(\{p\}\{q\})^\omega$.

(d) $\neg\mathbf{F}p \to \mathbf{F}\neg\mathbf{F}p$ is a tautology since $\varphi \to \mathbf{F}\varphi$ is a tautology for every formula φ.

(e) $(\mathbf{G}p \to \mathbf{F}q) \leftrightarrow (p \mathbf{U} (\neg p \vee q))$ is a tautology. We have

$$\begin{aligned}
\mathbf{G}p \to \mathbf{F}q &\equiv \neg\mathbf{G}p \vee \mathbf{F}q && \text{(by def. of implication)} \\
&\equiv \mathbf{F}\neg p \vee \mathbf{F}q && \text{(by \#13.1c)} \\
&\equiv \mathbf{F}(\neg p \vee q) && \text{(by \#13.3d)} \\
&\equiv \mathbf{F}(p \to q) && \text{(by def. of implication)}
\end{aligned}$$

Therefore, we have to show that

$$\mathbf{F}(p \to q) \leftrightarrow (p \mathbf{U} (p \to q)).$$

$\leftarrow$) Let σ be such that $\sigma \models (p \mathbf{U} (p \to q))$. In particular, there exists $k \geq 0$ such that $\sigma^k \models (p \to q)$. Therefore, $\sigma \models \mathbf{F}(p \to q)$.

$\to$) Let σ be such that $\sigma \models \mathbf{F}(p \to q)$. Let $k \geq 0$ be the smallest position such that $\sigma^k \models (p \to q)$. For every $0 \leq i < k$, we have $\sigma^i \not\models (p \to q)$ which is equivalent to $\sigma^i \models p \wedge \neg q$. Therefore, for every $0 \leq i < k$, we have $\sigma^i \models p$. This implies that $\sigma \models p \mathbf{U} (p \to q)$.

(f) $\neg(p \mathbf{U} q) \leftrightarrow (\neg p \mathbf{U} \neg q)$ is not a tautology. Let $\sigma = \emptyset\{q\}^\omega$. We have $\sigma \models \neg(p \mathbf{U} q)$ and $\sigma \not\models (\neg p \mathbf{U} \neg q)$.

(g) $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$ is a tautology since

$$\begin{aligned}
\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p) &\equiv \neg\mathbf{G}(\neg p \vee \mathbf{X}p) \vee (\neg p \vee \mathbf{G}p) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p && \text{(by \#13.1c)} \\
&\equiv \neg\mathbf{G}p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}\neg p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) && \text{(by \#13.1c)} \\
&\equiv \mathbf{F}\neg p \to \mathbf{F}\neg p.
\end{aligned}$$

Solution 13.5

(a) $(p \wedge q) \wedge \mathbf{X}(\neg p \wedge \neg q)$

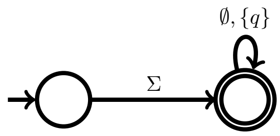
(b) $\mathbf{FG}(\neg p \wedge q)$

(c) $\mathbf{F}(p \wedge \mathbf{XF}(\neg p \wedge q))$

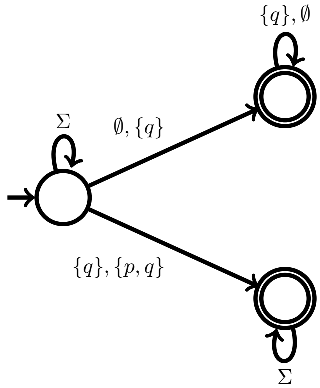
(d) $(p \wedge \neg q)\ \mathbf{U}\ ((\neg p \wedge q)\ \mathbf{U}\ \mathbf{G}(\neg p \wedge \neg q))$

Solution 13.6

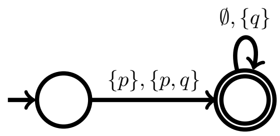
(a)



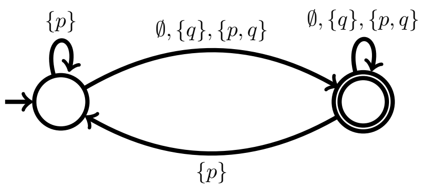
(b) Note that $(\mathbf{GF}p) \rightarrow (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \vee (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \vee (\mathbf{F}q)$. We build Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



(c) Note that $p \wedge \neg(\mathbf{XF}p) \equiv p \wedge \mathbf{XG}\neg p$. We build a Büchi automaton for $p \wedge \mathbf{XG}\neg p$:



(d)



(e)