Automata and Formal Languages — Homework 11

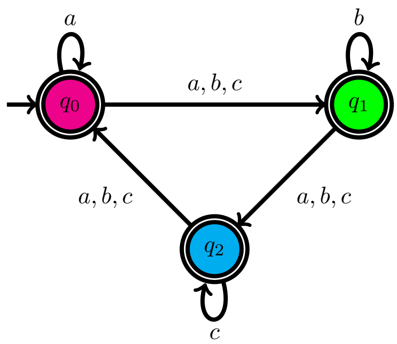
Due 20.01.2017

Exercise 11.1

(a) Give a deterministic Rabin automaton for $L = \{w \in \{a, b\}^\omega : w \text{ contains finitely many } a\text{'s}\}$.

(b) Give a Büchi automaton for L and try to "determinize" it by using the NFA to DFA powerset construction.

(c) What ω-language is accepted by the following Muller automaton with acceptance condition $\{\{q_0\}, \{q_1\}, \{q_2\}\}$?



(d) Show that any Büchi automaton that accepts the ω-language of (c) has more than 3 states.

(e) For every $m, n \in \mathbb{N}_{>0}$, let $L_{m,n}$ be the ω-language over $\{a, b\}$ described by $(a + b)^*((a^m bb)^\omega + (a^n bb)^\omega)$.

(i) Describe a family of Büchi automata accepting the family of ω-languages $\{L_{m,n}\}_{m,n \in \mathbb{N}_{>0}}$.

(ii) Show that there exists $c \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}_{>0}$, $L_{m,n}$ is accepted by a Rabin automaton with at most $\max(m, n) + c$ states.

(iii) Modify your construction in (ii) to obtain Muller automata instead of Rabin automata.

(iv) Convert the Rabin automaton obtained in (ii) for $L_{m,n}$ into a Büchi automaton.

Exercise 11.2

(a) Give deterministic Büchi automata for L_a, L_b, L_c where $L_\sigma = \{w \in \{a, b, c\}^\omega : w \text{ contains infinitely many } \sigma\text{'s}\}$, and build the intersection of these automata.

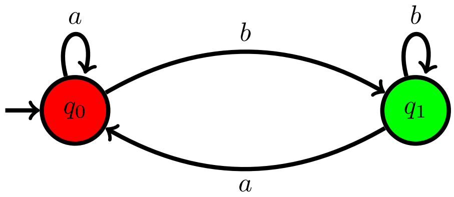
(b) Give Büchi automata for the following ω-languages:

- $L_1 = \{w \in \{a, b\}^\omega : w \text{ contains infinitely many } a\text{'s}\}$,
- $L_2 = \{w \in \{a, b\}^\omega : w \text{ contains finitely many } b\text{'s}\}$,
- $L_3 = \{w \in \{a, b\}^\omega : \text{each occurrence of } a \text{ in } w \text{ is followed by a } b\}$,

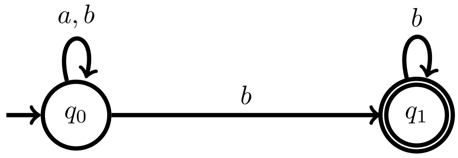
and build the intersection of these automata.

Solution 11.1

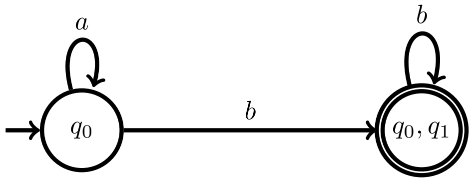
(a) The following Rabin automaton with acceptance condition $\{(\{q_1\}, \{q_0\})\}$, i.e. where q_1 must be visited infinitely often and q_0 must be visited finitely often:



(b) This Büchi automaton accepts L:



However, the powerset construction yields the following Büchi automaton which does not accept L since it does not accept bab^ω:



(c) $\Sigma^*(a^\omega + b^\omega + c^\omega)$.

(d) Assume there exists a Büchi automaton $B = (Q, \Sigma, \delta, Q_0, F)$ such that $|Q| \leq 3$ and $L_\omega(B)$ is the ω-language of (c). Let $w_\sigma = abc\sigma^\omega$. Since $w_a, w_b, w_c \in L_\omega(B)$, a pigeonhole argument shows that there exist $p_1, p_2, p_3 \in Q_0$, $q_1, q_2, q_3 \in F$, $m_1, m_2, m_3 \in \mathbb{N}$ and $n_1, n_2, n_3 \in \mathbb{N}_{>0}$ such that

$$\begin{aligned} p_1 &\xrightarrow{abca^{m_1}} q_1 \xrightarrow{a^{n_1}} q_1, \\ p_2 &\xrightarrow{abcb^{m_2}} q_2 \xrightarrow{b^{n_2}} q_2, \\ p_3 &\xrightarrow{abcc^{m_3}} q_3 \xrightarrow{c^{n_3}} q_3. \end{aligned}$$

We must have $q_i \neq q_j$ for every $i \neq j$, otherwise we would obtain a contradiction. For example, if $q_1 = q_2$, we have

$$p_1 \xrightarrow{abca^{m_1}} q_2 \xrightarrow{a^{n_1}} q_2 \xrightarrow{b^{n_2}} q_2 \xrightarrow{a^{n_1}} \cdots$$

which is a contradiction since $abca^{m_1}(a^{n_1}b^{n_2})^\omega \notin L_\omega(B)$. Therefore, $|Q| = 3$ and $Q = \{q_1, q_2, q_3\}$.

We have $p_i \neq q_i$ for every $i \in [3]$, otherwise we would again obtain a contradiction. For example, if $p_1 = q_1$, we have

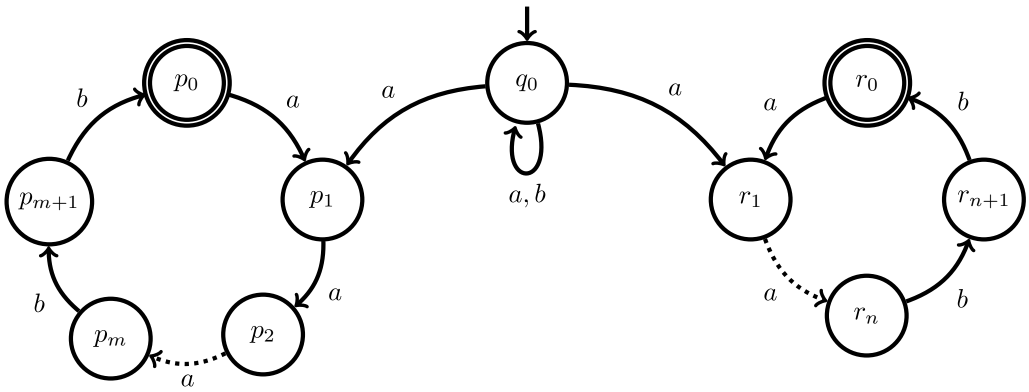
$$p_1 \xrightarrow{abca^{m_1}} p_1 \xrightarrow{abca^{m_1}} \cdots$$

which is a contradiction since $(abca^{m_1})^\omega \notin L_\omega(B)$.

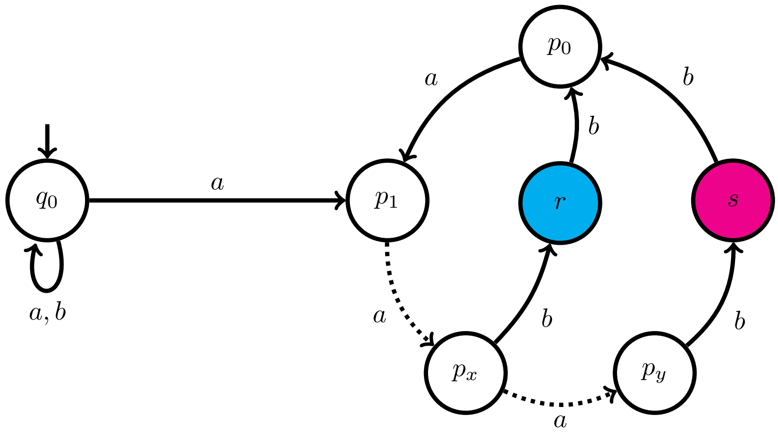
Suppose $p_1 = q_2$. If $p_2 = q_1$, then $(abca^{m_1}abcb^{m_2})^\omega \in L_\omega(B)$, which is a contradiction. Hence, $p_2 = q_3$. If $p_3 = q_2$, then $(abca^{m_2}abcb^{m_3})^\omega \in L_\omega(B)$, which is a contradiction. Hence, $p_3 = q_1$. This also yields a contradiction since it implies $(abca^{n_1}abcc^{n_3}abcb^{n_2})^\omega$. We conclude that $p_1 \neq q_2$.

A similar argument shows that $p_1 \neq q_3$, which contradicts $|Q| = 3$.

(e) (i)



(ii) Let $m, n \in \mathbb{N}_{>0}$. Let $x = \min(m, n)$ and $y = \max(m, n)$. We build the following Rabin automaton $B_{m,n}$:

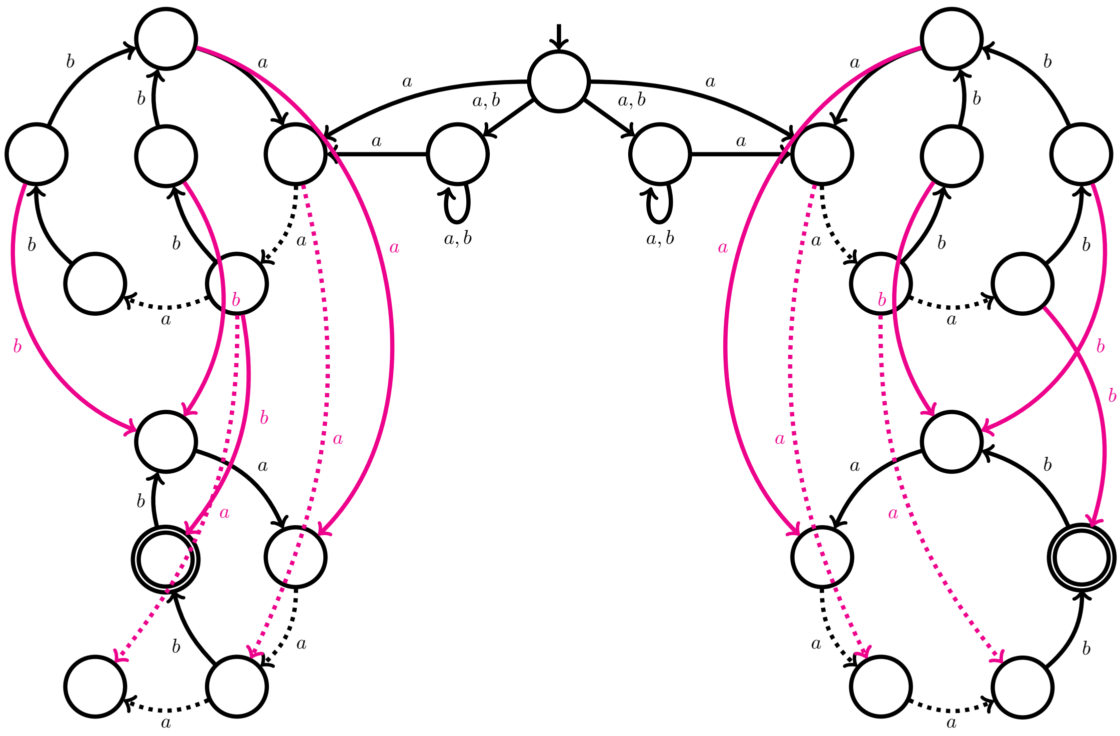


$B_{m,n}$ accepts $L_{m,n}$ when taking the acceptance condition $\{(\{r\}, \{s\}), (\{s\}, \{r\})\}$. Moreover, $B_{m,n}$ has $\max(m, n) + 3$ states.

(iii) We keep the same automaton $B_{m,n}$, but we change the acceptance condition to:

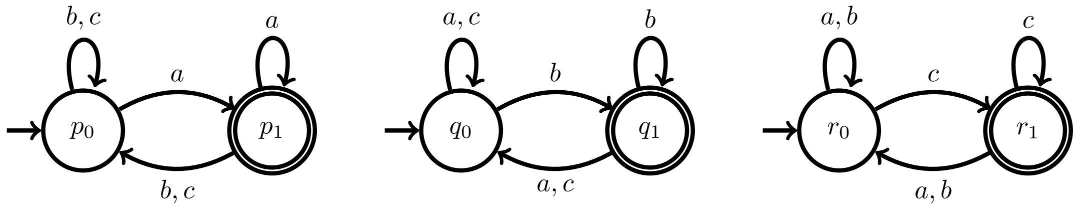
$$\{\{p_0, p_1, \ldots, p_x, r\}, \{p_0, p_1, \ldots, p_y, s\}\}.$$

(iv) The following Büchi automaton $C_{m,n}$ is obtained from the conversion of $B_{m,n}$:

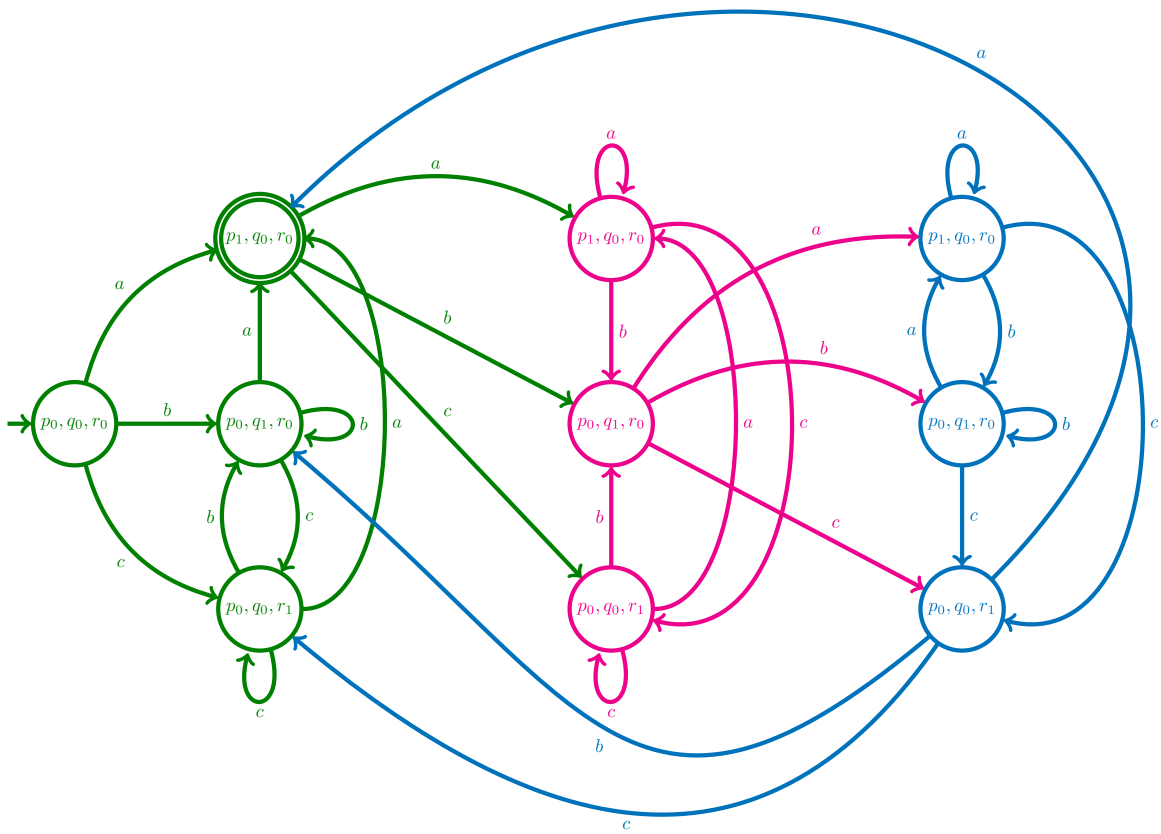


Solution 11.2

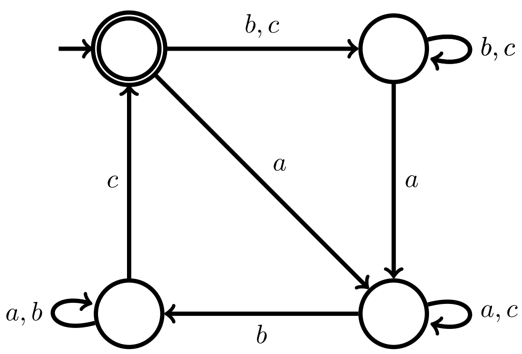
(a) The following deterministic Büchi automata respectively accept L_a, L_b and L_c:



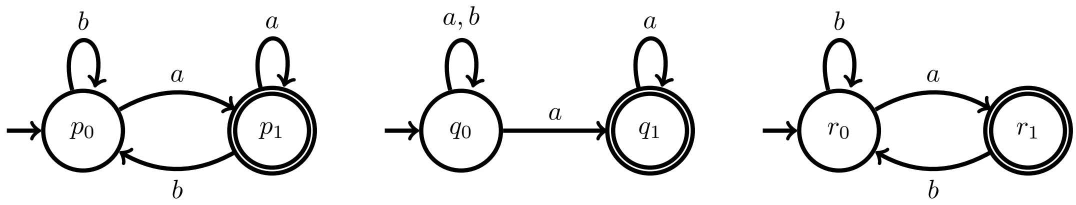
Taking the intersection of these automata leads to the following deterministic Büchi automaton:



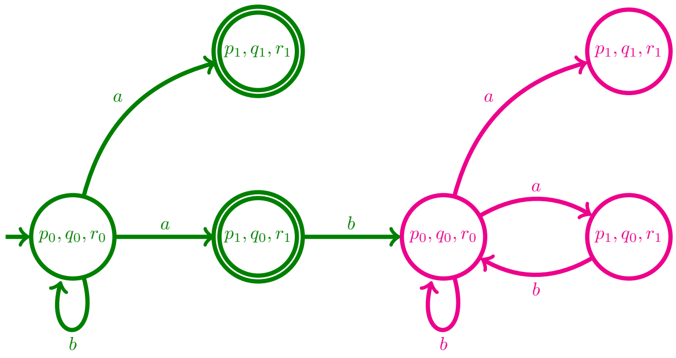
As seen in #10.1(c), $L_a \cap L_b \cap L_b$ is accepted by a smaller deterministic Büchi automaton:



(b) The following Büchi automata respectively accept L_1, L_2 and L_3:



Taking the intersection of these automata leads to the following Büchi automaton:



Note that this automaton accepts $\emptyset$.