Technische Universität München **Winter term 2016/17**
I7
Prof. J. Esparza / Dr. M. Blondin **08.01.2017**

Automata and Formal Languages — Homework 10

Due 13.01.2017

Exercise 10.1
Let $\inf(w)$ denote the set of letters occurring infinitely often in the infinite word w. Give Büchi automata and ω-regular expressions for the following ω-languages over $\Sigma = \{a, b, c\}$:

(a) $L_1 = \{w \in \Sigma^\omega : \inf(w) \subseteq \{a, b\}\}$,

(b) $L_2 = \{w \in \Sigma^\omega : \inf(w) = \{a, b\}$,

(c) $L_3 = \{w \in \Sigma^\omega : \{a, b\} \subseteq \inf(w)\}$,

(d) $L_4 = \{w \in \Sigma^\omega : \inf(w) = \{a, b, c\}\}$.

Exercise 10.2
Give deterministic Büchi automata accepting the following ω-languages over $\Sigma = \{a, b, c\}$:

(a) $L_1 = \{w \in \Sigma^\omega : w$ contains at least one $c\}$,

(b) $L_2 = \{w \in \Sigma^\omega :$ in w, every a is immediately followed by a $b\}$,

(c) $L_3 = \{w \in \Sigma^\omega :$ in w, between two successive a's there are at least two b's$\}$.

Exercise 10.3
Prove or disprove:

(a) For every Büchi automaton A, there exists a Büchi automaton B with a single initial state and such that $L_\omega(A) = L_\omega(B)$;

(b) For every Büchi automaton A, there exists a Büchi automaton B with a single accepting state and such that $L_\omega(A) = L_\omega(B)$;

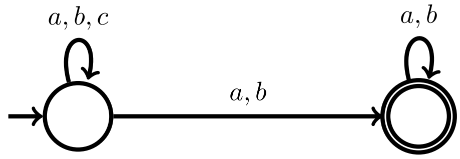
(c) Every finite ω-language is accepted by a Büchi automaton.

Exercise 10.4
Consider the class of non deterministic automata over infinite words with the following acceptance condition: an infinite run is accepting if it visits a final state *at least once*. Show that no such automaton accepts

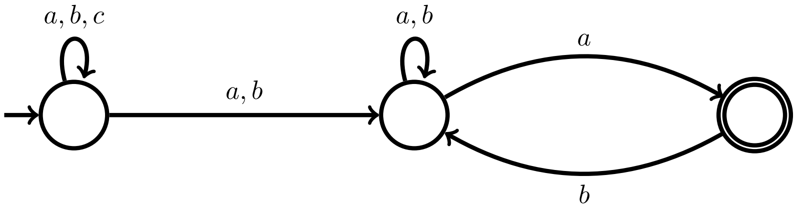
$$L = \{w \in \{a, b\}^\omega : w \text{ has infinitely many } a\text{'s and } b\text{'s}\}.$$

Solution 10.1

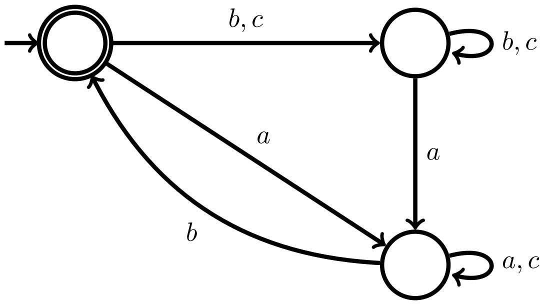
(a) $(a+b+c)^*(a+b)^\omega$, and



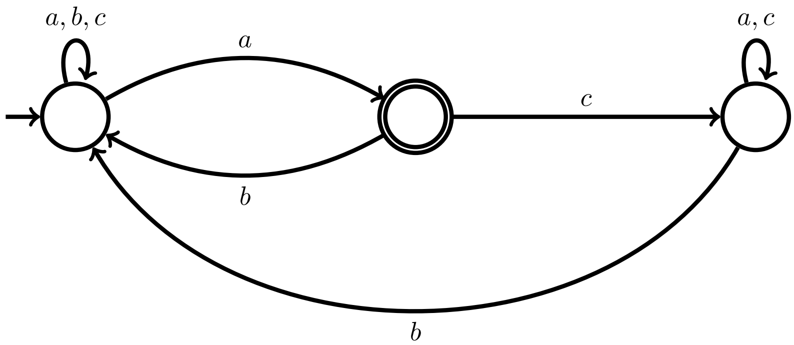
(b) $(a+b+c)^*(aa^*bb^*)^\omega$, and



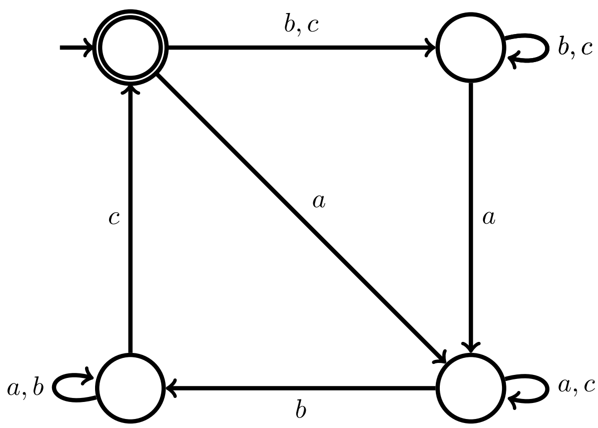
(c) $((b+c)^*a(a+c)^*b)^\omega$, and



or

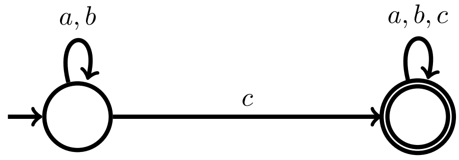


(d) $((b+c)^*a(a+c)^*b(a+b)^*c)^\omega$, and

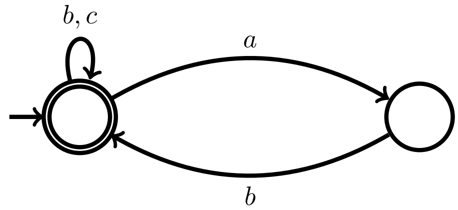


Solution 10.2

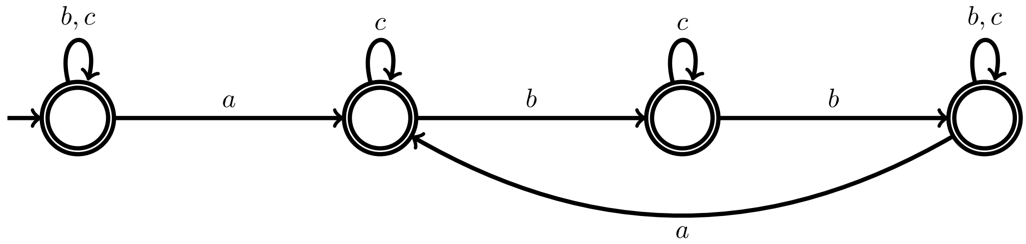
(a)



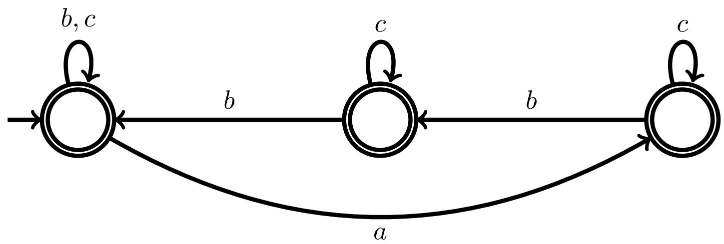
(b)



(c)



or simply,



Solution 10.3

(a) True. The construction for NFAs still work for Büchi automata.

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton. We add a state to Q which acts as the single initial state. More formally, we define $B' = (Q \cup \{q_{\text{init}}\}, \Sigma, \delta', \{q_{\text{init}}\}, F)$ where

$$\delta'(q, a) = \begin{cases} \bigcup_{q_0 \in Q_0} \delta(q_0, a) & \text{if } q = q_{\text{init}}, \\ \delta(q, a) & \text{otherwise.} \end{cases}$$

We have $L_\omega(B) = L_\omega(B')$, since there exists $q_0 \in Q_0$ such that

$$q_0 \xrightarrow{a_1}_B q_1 \xrightarrow{a_2}_B q_2 \xrightarrow{a_3}_B \cdots$$

if and only if

$$q_{\text{init}} \xrightarrow{a_1}_{B'} q_1 \xrightarrow{a_2}_{B'} q_2 \xrightarrow{a_3}_{B'} \cdots$$

(b) False. Let $L = \{a^\omega, b^\omega\}$. Suppose there exists a Büchi automaton $B = (Q, \{a, b\}, \delta, Q_0, \{q\})$ such that $L_\omega(B) = L$. Since $a^\omega \in L$, there exist $q_0 \in Q_0$, $m \geq 0$ and $n > 0$ such that

$$q_0 \xrightarrow{a^m} q \xrightarrow{a^n} q.$$

Similarly, since $b^\omega \in L$, there exist $q_0' \in Q_0$, $m' \geq 0$ and $n' > 0$ such that

$$q_0' \xrightarrow{b^{m'}} q \xrightarrow{b^{n'}} q.$$

This implies that

$$q_0 \xrightarrow{a^m} q \xrightarrow{b^{n'}} q \xrightarrow{b^{n'}} \cdots$$

Therefore, $a^m (b^{n'})^\omega \in L$, which is a contradiction. □

(c) False. Let $w \in \{0, 1\}^\omega$ be such that

$$w_i = \begin{cases} 1 & \text{if } i \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose there exists a Büchi automaton $B = (Q, \{0, 1\}, \delta, Q_0, F)$ such that $L_\omega(B) = \{w\}$. There exist $u \in \{0, 1\}^*$, $v \in \{0, 1\}^+$, $q_0 \in Q_0$ and $q \in F$ such that

$$q_0 \xrightarrow{u} q \xrightarrow{v} q.$$

Therefore, $uv^\omega \in L_\omega(B)$ which implies that $w = uv^\omega$. If $v \in 0^*$, then we obtain a contradiction. Thus, there exists $1 \leq i \leq |v|$ such that $v_i = 1$. Let $m = |u| + i$ and $n = |v|$. By definition of w, $m + j \cdot n$ is a square for every $j \geq 0$. In particular, there exist $0 < a < b$ such that

$$\begin{aligned} m + n \cdot n &= a^2, \\ m + n \cdot n + n &= b^2. \end{aligned}$$

Note that $a \geq n$. Moreover,

$$\begin{aligned} b^2 &= a^2 + n \\ &\leq a^2 + a \\ &< a^2 + 2a + 1 \\ &= (a+1)^2. \end{aligned}$$

Therefore $a^2 < b^2 < (a+1)^2$ which is a contradiction. □

Solution 10.4

Suppose there exists such an automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ accepting L. Since $w = ab^{|Q|}ab^{|Q|}\cdots$ belongs to L, there exist $u, v \in \{a, b\}^*$, $q_0 \in Q_0$, $q_{\text{acc}} \in F$, $q_0, q_1, \ldots q_{|Q|} \in Q$ such that

$$q_0 \xrightarrow{u} q_{\text{acc}} \xrightarrow{v} q_0 \xrightarrow{b} q_1 \xrightarrow{b} \cdots \xrightarrow{b} q_{|Q|}$$

By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Therefore,

$$q_0 \xrightarrow{u} q_{\text{acc}} \xrightarrow{vb^i} q_i \xrightarrow{b^{j-i}} q_j \xrightarrow{b^{j-i}} q_j \xrightarrow{b^{j-i}} \cdots$$

We conclude that $uvb^i(b^{j-i})^\omega$ is accepted by B, which is a contradiction.