Technische Universität München **Winter term 2016/17**
I7
Prof. J. Esparza / Dr. M. Blondin **18.12.2016**

Automata and Formal Languages — Homework 9

Due 23.12.2016

Exercise 9.1

(a) Give an MSO formula Block_between such that Block_between(X, i, j) holds whenever $X = \{i, i+1, \dots, j\}$.

(b) Let $0 \leq m < n$. Give an MSO formula $\text{Mod}^{m,n}$ such that $\text{Mod}^{m,n}(i, j)$ holds whenever $|w_i w_{i+1} \cdots w_j| \equiv m \pmod{n}$, i.e. whenever $j - i + 1 \equiv m \pmod{n}$.

(c) Let $0 \leq m < n$. Give an MSO sentence for $a^m(a^n)^*$.

(d) Give an MSO sentence for the language of words such that every two b's with no other b in between are separated by a block of a's of odd length.

Exercise 9.2

Let $r \geq 0, n \geq 1$. Give a Presburger formula φ such that $\mathcal{J} \models \varphi$ if, and only if, $\mathcal{J}(x) \geq \mathcal{J}(y)$ and $\mathcal{J}(x) - \mathcal{J}(y) \equiv r \pmod{n}$. Give an automaton that accepts the solutions of φ for $r = 0$ and $n = 2$.

Exercise 9.3

Use algorithms seen in class and algorithm *EqtoDFA* (p. 207) to build an automaton for $\exists y\ x = 3y$.

Exercise 9.4

Consider the extension of FO(Σ) where addition of variables is allowed. Give a sentence of this logic for palindromes over $\{a, b\}$, i.e. $\{w \in \{a, b\}^* : w = w^R\}$.

Solution 9.1

★ Note that questions (b) and (c) were updated in light of the discussions we had in class.

(a) $\text{Block\_between}(X, i, j) = \forall x\ (x \in X) \leftrightarrow (i \leq x \wedge x \leq j)$.

(b) $\text{Mod}^{m,n}(i, j) = \exists x\ (x = i + m) \wedge \text{Mult}^n(x, j)$ where

$$\text{Mult}^n(i, j) = \exists X\ (j \in X) \wedge (\forall x \in X\ [(x = i + n - 1) \vee \exists y \in X\ (y = x + n)])$$

(c) $[(m = 0) \wedge (\neg \exists x \text{ first}(x))] \vee [\forall x\ Q_a(x) \wedge \exists x, y \text{ first}(x) \wedge \text{last}(y) \wedge \text{Mod}^{m,n}(x, y)]$.

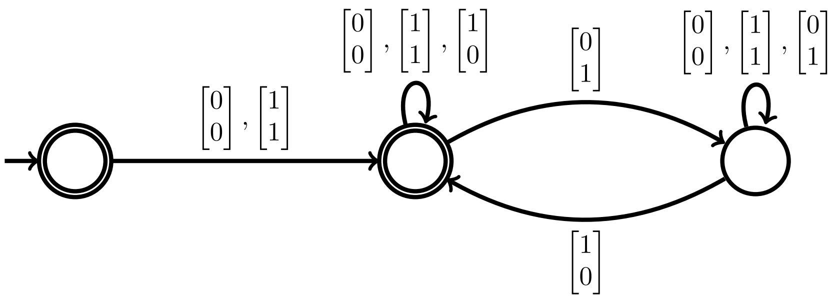
(d)

$$\forall x, y\ [(x < y) \wedge Q_b(x) \wedge Q_b(y) \wedge \forall z(x < z < y \wedge \neg Q_b(z))] \rightarrow$$
$$[(\forall z\ (x < z < y) \wedge Q_a(z)) \wedge (\exists x', y'\ (x' = x + 1) \wedge (y = y' + 1) \wedge \text{Mod}^{1,2}(x', y'))]\ .$$

Solution 9.2

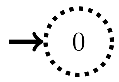
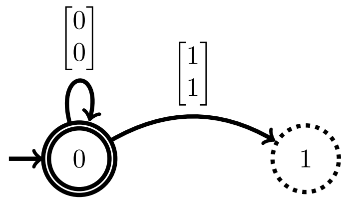
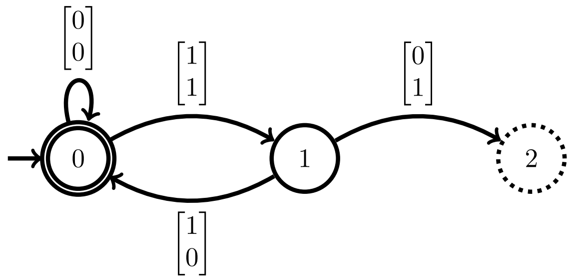
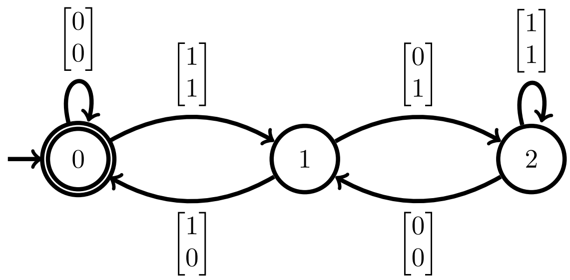
$$(x \leq y) \wedge (\exists a, b \bigvee_{0 \leq r' < n} (x - y = n \cdot a + r') \wedge (r = n \cdot b + r'))$$

Let $k \in \mathbb{N}$ and $x, y \in \Sigma^k$. First note that $\text{val}(x) - \text{val}(y) \equiv 0 \pmod 2$ if, and only if, $\text{val}(x)$ and $\text{val}(y)$ are either both odd or both even. Thus, the first bit of x and y should be the same. Moreover, $\text{val}(x) \geq \text{val}(y)$ if, and only if, $x = y$ or if there exists $\ell \in [k]$ such that $x_\ell = 1$, $y_\ell = 0$, and $x_i \geq y_i$ for every $\ell < i \leq k$. These observations yield the following automaton:

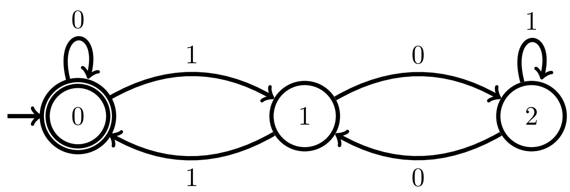


Solution 9.3

We can rewrite the formula as $\exists y\ x - 3y = 0$. We first use *EqtoDFA* to obtain an automaton for $x - 3y = 0$:

Iter.	**Current automaton**	W
0		$\{0\}$
1		$\{1\}$
2		$\{2\}$
3		$\emptyset$

It remains to project the automaton on x, i.e. on the first component of the letters. We obtain:



Solution 9.4

$$(\neg \exists x \text{ first}(x)) \vee (\exists x, y \text{ first}(x) \wedge \text{last}(y) \wedge \bigvee_{a \in \Sigma} (Q_a(x) \wedge Q_a(y)) \wedge$$
$$[\forall x', y', \ell \ (x' = x + \ell \wedge y = y' + \ell) \rightarrow \bigvee_{a \in \Sigma} (Q_a(x') \wedge Q_a(y'))]) \ .$$