Technische Universität München **Winter term 2016/17**
I7
Prof. J. Esparza / Dr. M. Blondin **18.10.2016**

Automata and Formal Languages — Homework 1

Due 21.10.2016

Exercise 1.1
Consider the language $L \subseteq \{a, b\}^*$ given by the regular expression $a^*b^*a^*a$.

(a) Give an NFA-ε that accepts L.

(b) Give an NFA that accepts L.

(c) Give a DFA that accepts L.

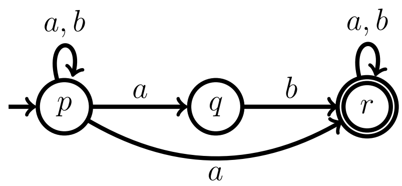
Exercise 1.2
Let $L = \{w \in \{a, b\}^* : w \text{ does not contain any occurrence of } aa\}$.

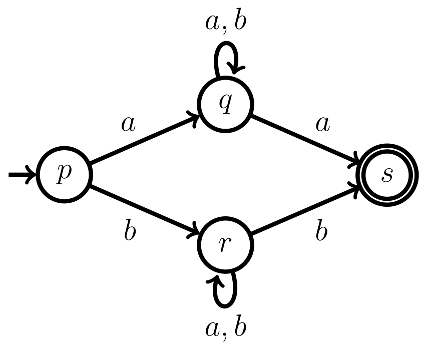
(a) Give a DFA that accepts L.

(b) Give a regular expression for L.

(c) Prove that the regular expression given in (b) is correct, i.e. that its language is indeed L.

Exercise 1.3
Consider the two following NFAs A and B working over alphabet $\{a, b\}$:





(a) Describe $L(A)$ and $L(B)$ in English.

(b) Give regular expressions for $L(A)$ and $L(B)$.

(c) Determinize A and B, i.e. convert A and B to DFAs accepting the same languages.

Exercise 1.4
Let A and B be DFAs over some alphabet Σ.

(a) Describe DFAs C and D such that $L(C) = L(A) \cup L(B)$ and $L(D) = L(A) \cap L(D)$.

(b) Prove that C is correct, i.e. that indeed $L(C) = L(A) \cup L(B)$.

(c) If A and B were NFAs, could you construct NFAs with fewer states for union and intersection? Explain your answer.

Exercise 1.5
The *reverse* of a word $w \in \Sigma^*$ is defined as $w^R = \varepsilon$ if $w = \varepsilon$ and $w^R = a_n a_{n-1} \cdots a_1$ if $w = a_1 a_2 \cdots a_n$. The *reverse* of a language $L \subseteq \Sigma^*$ is defined as $L^R = \{w^R : w \in L\}$.

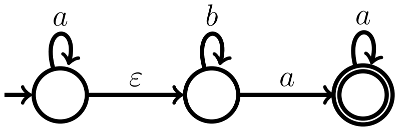
Let A be an NFA. Describe an NFA B such that $L(B) = L(A)^R$.

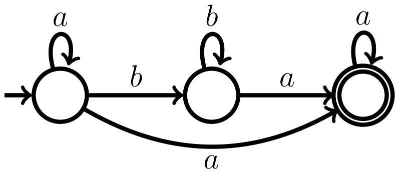
Exercise 1.6
The *shuffle* of two languages $A, B \subseteq \Sigma^*$ is defined as

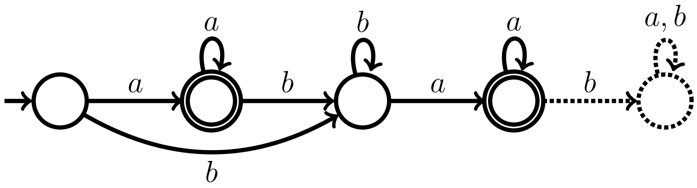
$$A \sqcup\!\sqcup B = \{w \in \Sigma^* : \exists a_1, \ldots, a_n, b_1, \ldots, b_n \in \Sigma^* \text{ s.t. } \begin{array}{l} a_1 \cdots a_n \in A \text{ and} \\ b_1 \cdots b_n \in B \text{ and} \\ w = a_1 b_1 \cdots a_n b_n \ \} \ . \end{array}$$

Let A and B be DFAs. Describe an NFA C such that $L(C) = L(A) \sqcup\!\sqcup L(B)$.

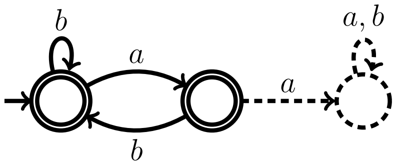
Solution 1.1

(a) 

(b) 

(c) 

Solution 1.2

(a) 

(b) $r = (a + \varepsilon)(b^* + ba)^*$ or $r' = (b^* + ab)^*(a + \varepsilon)$

(c) <u>$L(r) \subseteq L$:</u>

Let $w \in L(r)$. By definition of r, $w = u_1 u_2 \cdots u_n$ for some $n \in \mathbb{N}$, $u_1 \in \{\varepsilon, a\}$ and $u_2, \ldots, u_n \in \{b^*, ba\}$. Assume r contains an occurrence of aa. Since none of the u_i contains aa, there must exist some $i \geq 0$ such that u_i ends with a and u_{i+1} starts with a. The only possible case for u_{i+1} is $u_{i+1} = a$, hence $i + 1 = 1$ and $i = 0$ which is a contradiction.

<u>$L \subseteq L(r)$:</u>

Let $w \in L$. There exist $n \in \mathbb{N}$ and $i, j_1, j_2, \ldots j_n, k \in \mathbb{N}$ such that

- $w = b^i a b^{j_1} a b^{j_2} \cdots a b^{j_n} a b^k$,
- $i, k \geq 0$,
- $j_1, j_2, \ldots, j_n > 0$.

If $i = 0$, we way derive w as follows:

a	b^*	ba	$\cdots$	b^*	ba	b^*
a	b^{j_1-1}	ba	$\cdots$	b^{j_n-1}	ba	b^k

If $i > 0$, we way derive w as follows:

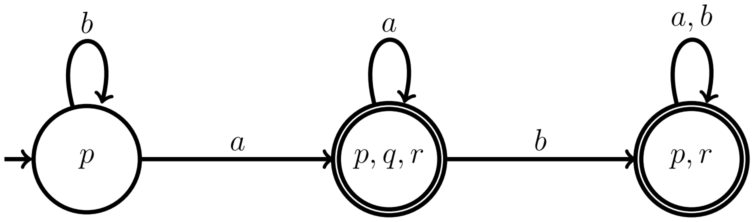
ε	b^*	ba	b^*	ba	$\cdots$	b^*	ba	b^*
	b^{i-1}	ba	b^{j_1-1}	ba	$\cdots$	b^{j_n-1}	ba	b^k

Solution 1.3

A) (a) $L(A) = \{w \in \{a,b\}^* : w \text{ contains } ab \text{ or } a\} = \{w \in \{a,b\}^* : w \text{ contains } a\}$,

(b) $(a+b)^*(ab+a)(a+b)^*$ or $(a+b)^*a(a+b)*$ or $b^*a(a+b)^*$,

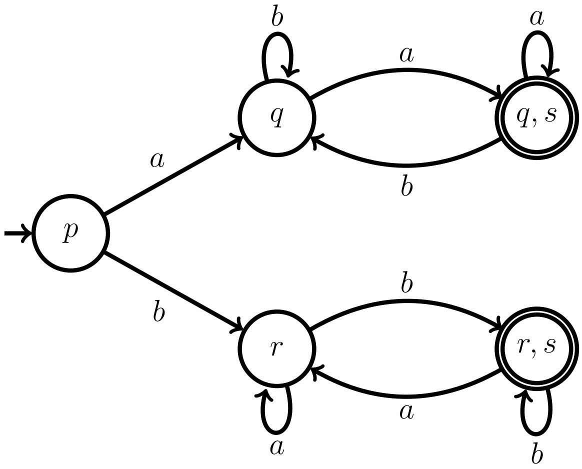
(c)



B) (a) $L(B) = \{w \in \{a,b\}^* : |w| > 1 \text{ and } w \text{ starts and ends with the same letter}\}$

(b) $a(a+b)^*a + b(a+b)^*b$

(c)



Solution 1.4

(a) Let $A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_0', F_B)$. We define C and D as follows:

$$C = (Q_A \times Q_B, \Sigma, \delta', (q_0, q_0'), F_C)$$
$$D = (Q_A \times Q_B, \Sigma, \delta', (q_0, q_0'), F_D)$$

where $\delta'((p,q),a) = (\delta_A(p,a), \delta_B(q,a))$ and

$$F_C = \{(p,q) \in Q_A \times Q_B : p \in F_A \vee q \in F_B\}$$
$$F_D = \{(p,q) \in Q_A \times Q_B : p \in F_A \wedge q \in F_B\}$$

(b) It suffices to prove that

$$(p,q) \xrightarrow{w}_C (p',q') \iff p \xrightarrow{w}_A p' \text{ and } q \xrightarrow{w}_B q' \ .$$

We proceed by induction on $|w|$. If $|w| = 0$, then $w = \varepsilon$ and the claim trivially holds. Assume that $|w| > 0$ and suppose that the claim holds for every word of length $|w| - 1$. Let $w = a_1 a_2 \cdots a_n$. We have,

$$\begin{aligned}
(p,q) \xrightarrow{w}_C (p',q') &\iff \delta'((p,q),a_1) = (p'',q'') && \text{and } (p'',q'') \xrightarrow{a_2\cdots a_n}_C (p',q') \\
&\iff \delta'((p,q),a_1) = (p'',q'') && \text{and } p'' \xrightarrow{a_2\cdots a_n}_A p' \text{ and } q'' \xrightarrow{a_2\cdots a_n}_B q' \text{ (by ind. hyp.)} \\
&\iff \delta_A(p,a_1) = p'' \text{ and } \delta_B(q,a_1) = q'' && \text{and } p'' \xrightarrow{a_2\cdots a_n}_A p' \text{ and } q'' \xrightarrow{a_2\cdots a_n}_B q' \text{ (by def. of } C) \\
&\iff p \xrightarrow{a_1 a_2\cdots a_n}_A p' \text{ and } q \xrightarrow{a_1 a_2\cdots a_n}_B q' \\
&\iff p \xrightarrow{w}_A p' \text{ and } q \xrightarrow{w}_B q' \ . && \square
\end{aligned}$$

(c) Intersection sometimes require the $|Q_A| \cdot |Q_B|$ states from the product construction [1, Thm. 11], however it is possible to do better with union. Since multiple initial states are allowed in this course, we can simply build the following NFA:

$$C = (Q_A \cup Q_B, \Sigma, \delta_A \cup \delta_B, Q_0 \cup Q_0', F_A \cup F_B) \ .$$

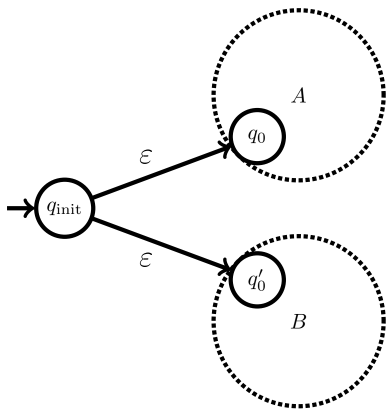
If we were restricted to a single initial state, we could build the following NFA:

$$C = (q_{\text{init}} \cup Q_A \cup Q_B, \Sigma, \delta', q_{\text{init}}, F_A \cup F_B)$$

where

$$\delta'(q, a) = \begin{cases} \delta_A(q_0, a) \cup \delta_B(q_0', a) & \text{if } q = q_{\text{init}}, \\ \delta_A(q, a) & \text{if } q \in Q_A, \\ \delta_B(q, a) & \text{if } q \in Q_B. \end{cases}$$

The last construction would even be simpler if ε-transitions were allowed:



Solution 1.5

We simply flip transitions of A and swap initial and final states. More formally, let $A = (Q, \Sigma, \delta_A, Q_0, F)$. We define B as $B = (Q, \Sigma, \delta_B, F, Q_0)$ where $\delta_B(p, a) = \{q \in Q : p \in \delta_A(q, a)\}$.

Solution 1.6

We give an NFA C that simulates A and B by "non deterministically guessing" in which automaton each letter must be read.

Let $A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_0', F_B)$. We let $C = (Q_A \times Q_B, \Sigma, \delta_C, (q_0, q_0'), F_A \times F_B)$ where

$$\delta_C((p, q), a) = \{(p', q) : p' \in \delta_A(p, a)\} \cup \{(p, q') : q' \in \delta_B(q, a)\} \ .$$

References

[1] Markus Holzer and Martin Kutrib. State complexity of basic operations on nondeterministic finite automata. In *Proc. 7^{th} International Conference on Implementation and Application of Automata (CIAA)*, pages 148–157, 2002.