Logic

Logics on words

- Regular expressions give operational descriptions of regular languages.
- Often the natural description of a language is declarative:
  - **even number of a's and even number of b's** vs.

    $$\big(aa + bb + (ab + ba)(aa + bb)^*(ba + ab)\big)^*$$
  - **words not containing 'hello'**
- Goal: find a declarative language able to express all the regular languages, and only the regular languages.

Logics on words

- Idea: use a logic that has an interpretation on words
- A formula expresses a property that each word may satisfy or not, like
  - **the word contains only a's**
  - **the word has even length**
  - **between every occurrence of an a and a b there is an occurrence of a c**
- Every formula (indirectly) defines a language: the language of all the words over the given fixed alphabet that satisfy it.

First-order logic on words

- Atomic formulas: for each letter a we introduce the formula $Q_a(x)$, with intuitive meaning: **the letter at position x is an a**.

First-order logic on words: Syntax

- Formulas constructed out of atomic formulas by means of standard “logic machinery”:
  - Alphabet $\Sigma = \{a, b, \ldots\}$ and position variables $V = \{x, y, \ldots\}$
  - $Q_a(x)$ is a formula for every $a \in \Sigma$ and $x \in V$.
  - $x < y$ is a formula for every $x, y \in V$
  - If $\varphi, \varphi_1, \varphi_2$ are formulas then so are $\neg\varphi$ and $\varphi_1 \vee \varphi_2$
  - If φ is a formula then so is $\exists x\ \varphi$ for every $x \in V$

Abbreviations

$$\varphi_1 \wedge \varphi_2 := \neg(\neg\varphi_1 \vee \neg\varphi_2)$$

$$\varphi_1 \rightarrow \varphi_2 := \neg\varphi_1 \vee \varphi_2$$

$$\forall x\, \varphi := \neg\exists\, x \neg\varphi$$

$$\begin{aligned} \text{first}(x) &:= \\ \text{last}(x) &:= \\ y = x + 1 &:= \\ y = x + 2 &:= \\ y = x + (k+1) &:= \end{aligned}$$

Examples (without semantics yet)

- "The last letter is a b and before it there are only a's."

$$\exists x\, Q_b(x) \wedge \forall x\, (\text{last}(x) \rightarrow Q_b(x) \ \wedge\ \neg\text{last}(x) \rightarrow Q_a(x))$$

- "Every a is immediately followed by a b."

$$\forall x\, (Q_a(x) \rightarrow \exists y\, (y = x + 1 \wedge Q_b(y)))$$

- "Every a is immediately followed by a b, unless it is the last letter."

$$\forall x\, (Q_a(x) \rightarrow \forall y\, (y = x + 1 \rightarrow Q_b(y)))$$

- "Between every a and every later b there is a c."

$$\forall x \forall y\, (Q_a(x) \wedge Q_b(y) \wedge x < y \rightarrow \exists z\, (x < z \wedge z < y \wedge Q_c(z)))$$

First-order logic on words: Semantics

- Formulas are interpreted on pairs $(w, \mathcal{J})$ called interpretations, where
  - w is a word, and
  - $\mathcal{J}$ assigns positions to the free variables of the formula (and maybe to others too—who cares)
- It does not make sense to say a formula is true or false: it can only be true or false for a given interpretation.
- If the formula has no free variables (if it is a sentence), then for each word it is either true or false.

- Satisfaction relation:

$$
\begin{array}{lcllr}
(w, \mathfrak{I}) & \models & Q_a(x) & \textit{iff} & w[\mathfrak{I}(x)] = a \\
(w, \mathfrak{I}) & \models & x < y & \textit{iff} & \mathfrak{I}(x) < \mathfrak{I}(y) \\
(w, \mathfrak{I}) & \models & \neg\varphi & \textit{iff} & (w, \mathfrak{I}) \not\models \varphi \\
(w, \mathfrak{I}) & \models & \varphi 1 \vee \varphi_2 & \textit{iff} & (w, \mathfrak{I}) \models \varphi_1 \textit{ or } (w, \mathfrak{I}) \models \varphi_2 \\
(w, \mathfrak{I}) & \models & \exists x\, \varphi & \textit{iff} & |w| \geq 1 \textit{ and some } i \in \{1, \ldots, |w|\} \textit{ satisfies } (w, \mathfrak{I}[i/x]) \models \varphi
\end{array}
$$

- More logic jargon:
  - A formula is valid if it is true for all its interpretations
  - A formula is satisfiable if is is true for at least one of its interpretations

The empty word ...

- ... is as usual a pain in the eh, neck.
- It satisfies all universally quantified formulas, and no existentially quantified formula.

Can we only express regular languages? Can we express all regular languages?

- The language $L(\varphi)$ of a sentence φ is the set of words that satisfy φ.
- A language L is expressible in first-order logic or FO-definable if some sentence φ satisfies $L(\varphi) = L$.
- Proposition: a language over a one-letter alphabet is expressible in first-order logic iff it is finite or co-finite (its complement is finite).
- Consequence: we can only express regular languages, but not all, not even the language of words of even length.

Proof sketch

1. If L is finite, then it is FO-definable

2. If L is co-finite, then it is FO-definable.

Proof sketch

3. If L is FO-definable (over a one-letter alphabet), then it is finite or co-finite.
   1) We define a new logic QF (quantifier-free fragment)
   2) We show that a language is QF-definable iff it is finite or co-finite
   3) We show that a language is QF-definable iff it FO-definable.

1) The logic QF

- $x < k$ $\quad x > k$

  $x < y + k$ $\quad x > y + k$

  $k < last$ $\quad k > last$

  are formulas for every variable x, y and every $k \geq 0$.
- If f_1, f_2 are formulas, then so are $f_1 \vee f_2$ and $f_1 \wedge f_2$

2) L is QF-definable iff it is finite or co-finite

(→) Let f be a sentence of QF.

Then f is an and-or combination of formulas $k < last$ and $k > last$.

$L(k < last) = \{k+1, k+2, \dots\}$ is co-finite (we identify words and numbers)

$L(k > last) = \{0,1,\dots,k\}$ is finite

$L(f_1 \vee f_2) = L(f_1) \cup L(f_2)$ and so if $L(f)$ and $L(g)$ finite or co-finite the L is finite or co-finite.

$L(f_1 \wedge f_2) = L(f_1) \cap L(f_2)$ and so if $L(f)$ and $L(g)$ finite or co-finite the L is finite or co-finite.

2) L is QF-definable iff it is finite or co-finite

($\leftarrow$) If $L = \{k_1, \ldots, k_n\}$ is finite, then

$$(k_1 - 1 < last \land last < k_1 + 1) \lor \cdots \lor$$
$$(k_n - 1 < last \land last < k_n + 1)$$

expresses L.

If L is co-finite, then its complement is finite, and so expressed by some formula. We show that for every f some formula $neg(f)$ expresses $\overline{L(f)}$

- $neg(k < last) = (k - 1 < last \land last < k + 1)$
  $\lor\ last < k$
- $neg(f_1 \lor f_2) = neg(f_1) \land neg(f_2)$
- $neg(f_1 \land f_2) = neg(f_1) \lor neg(f_2)$

3) Every first-order formula φ has an equivalent QF-formula $QF(\varphi)$

- $QF(x < y) = x < y + 0$
- $QF(\neg\varphi) = neg\big(QF(\varphi)\big)$
- $QF(\varphi_1 \vee \varphi_2) = QF(\varphi_1) \vee QF(\varphi_2)$
- $QF(\varphi_1 \wedge \varphi_2) = QF(\varphi_1) \wedge QF(\varphi_2)$
- $QF(\exists x\, \varphi) = QF(\exists x\, QF(\varphi))$
  - If $QF(\varphi)$ disjunction, apply $\exists x\, (\varphi_1 \vee \dots \vee \varphi_n) = \exists x\, \varphi_1 \vee \dots \vee \exists x\, \varphi_n$
  - If $QF(\varphi)$ conjunction (or atomic formula), see example in the next slide.

- Consider the formula

$$\exists x \quad x < y + 3 \quad \wedge$$
$$z < x + 4 \quad \wedge$$
$$z < y + 2 \quad \wedge$$
$$y < x + 1$$

- The equivalent QF-formula is

$$z < y + 8 \ \wedge \ y < y + 5 \ \wedge \ z < y + 2$$

Monadic second-order logic

- First-order variables: interpreted on positions
- Monadic second-order variables: interpreted on sets of positions.
  - Diadic second-order variables: interpreted on relations over positions
  - Monadic third-order variables: interpreted on sets of sets of positions
  - New atomic formulas: $x \in X$

Expressing „even length“

- Express

  There is a set X of positions such that

  – X contains exactly the even positions, and

  – the last position belongs to X.

- Express

  X contains exactly the even positions

  as

A position is in X iff it is second position or the second successor of another position of X

Syntax and semantics of MSO

- New set $\{X, Y, Z, \ldots\}$ of second-order variables
- New syntax: $x \in X$ and $\exists x\ \varphi$
- New semantics:
  - Interpretations now also assign sets of positions to the free second-order variables.
  - Satisfaction defined as expected.

Expressing $c^*(ab)^*d^*$

- Express:

There is a block X of consecutive positions such that

  - **before X there are only c's;**
  - **after X there are only b's;**
  - **a's and b's alternate in X;**
  - **the first letter in X is an a, and the last is a b.**

- Then we can take the formula

$$\exists X\ (Cons(X)\ \wedge Boc(X) \wedge Aod(X) \wedge \mathrm{Alt}(\mathrm{X}) \wedge Fa(X) \wedge Lb(X)\ )$$

- X is a block of consecutive positions

- Before X there are only c's

- In X a's and b's alternate

Every regular language is expressible in MSO logic

- Goal: given an arbitrary regular language L, construct an MSO sentence φ such having $L = L(\varphi)$.
- We use: if L is regular, then there is a DFA A recognizing L.
- Idea: construct a formula expressing

the run of A on this word is accepting

- Fix a regular language L.
- Fix a DFA A with states $q_0, \ldots, q_n$ recognizing L.
- Fix a word $w = a_1 a_2 \ldots a_m$.
- Let P_q be the set of positions i such that after reading $a_1 a_2 \ldots a_i$ the automaton A is in state q.
- We have:

A accepts w iff $m \in P_q$ for some final state q.

- Assume we can construct a formula

$$Visits(X_0, \dots, X_n)$$

which is true for $(w, \mathcal{J})$ iff

$$\mathcal{J}(X_0) = P_{q_0}, \dots, \mathcal{J}(X_n) = P_{q_n}$$

- Then $(w, \mathcal{J})$ satisfies the formula

$$\psi_A := \exists X_0 \dots \exists X_n \text{ Visits}(X_0, \dots X_n) \wedge \exists x \left(\text{last}(x) \wedge \bigvee_{q_i \in F} x \in X_i \right)$$

iff w has a last letter and $w \in L$, and we easily get a formula expressing L .

- To construct $Visits(X_0, \ldots, X_n)$ we observe that the sets P_q are the unique sets satisfying
  a) $1 \in P_{\delta(q_0, a_1)}$ i.e., after reading the first letter the DFA is in state $\delta(q_0, a_1)$.
  b) The sets P_q build a partition of the set of positions, i.e., the DFA is always in exactly one state.
  c) If $i \in P_q$ and $\delta(q, a_{i+1}) = q'$ then $i + 1 \in P_{q'}$, i.e., the sets „match“ δ.
- We give formulas for a) , b), and c)

- Formula for a)

$$\mathrm{Init}(X_0,\ldots,X_n) = \exists x \left(\mathrm{first}(x) \wedge \left(\bigvee_{a\in\Sigma}(Q_a(x) \wedge x \in X_{i_a})\right)\right)$$

- Formula for b)

$$\mathrm{Partition}(X_0,\ldots,X_n) = \forall x \left(\bigvee_{i=0}^{n} x \in X_i \;\wedge \bigwedge_{\substack{i,\, j = 0 \\ i \neq j}}^{n} (x \in X_i \rightarrow x \notin X_j)\right)$$

- Formula for c)

$$\mathrm{Respect}(X_0, \ldots, X_n) =$$

$$\forall x \forall y \left(y = x + 1 \rightarrow \bigvee_{\substack{a \in \Sigma \\ i, j \in \{0, \ldots, n\} \\ \delta(q_i, a) = q_j}} (x \in X_i \wedge Q_a(x) \wedge y \in X_j) \right)$$

- Together:

$$\begin{aligned} \mathrm{Visits}(X_0, \ldots X_n) := {} & \mathrm{Init}(X_0, \ldots, X_n) \wedge \\ & \mathrm{Partition}(X_0, \ldots, X_n) \wedge \\ & \mathrm{Respect}(X_0, \ldots, X_n) \end{aligned}$$

Every language expressible in MSO logic is regular

- Recall: an interpretation of a formula is a pair $(w, \mathcal{I})$ consisting of a word w and assignments $\mathcal{I}$ to the free first and second order variables (and perhaps to others).

$$\left(aab\ ,\ \begin{array}{l} x \mapsto 1 \\ y \mapsto 3 \\ X \mapsto \{2,3\} \\ Y \mapsto \{1,2\} \end{array} \right) \qquad \left(ba\ ,\ \begin{array}{l} x \mapsto 2 \\ y \mapsto 1 \\ X \mapsto \emptyset \\ Y \mapsto \{1\} \end{array} \right)$$

- We encode interpretations as words.

$$\left(aab\ ,\ \begin{array}{l} x \mapsto 1 \\ y \mapsto 3 \\ X \mapsto \{2,3\} \\ Y \mapsto \{1,2\} \end{array} \right) \qquad \left(ba\ ,\ \begin{array}{l} x \mapsto 2 \\ y \mapsto 1 \\ X \mapsto \emptyset \\ Y \mapsto \{1\} \end{array} \right)$$

	a	a	b
x	1	0	0
y	0	0	1
X	0	1	1
Y	1	1	0

	b	a
x	0	1
y	1	0
X	0	0
Y	1	0

- Given a formula with n free variables, we encode an interpretation $(w, \mathcal{I})$ as a word $enc(w, \mathcal{I})$ over the alphabet $\Sigma \times \{0,1\}^n$.
- The language of the formula φ , denoted by $L(\varphi)$, is given by

$$L(\varphi)=\{enc(w, \mathcal{I}) \mid (w, \mathcal{I}) \vDash \varphi\}$$

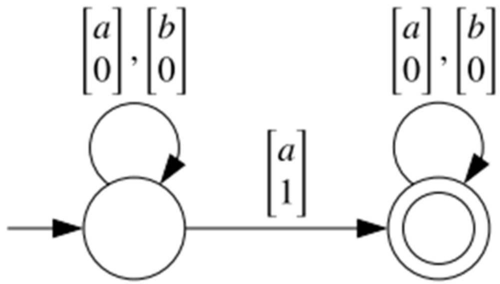
- We prove by induction on the structure of φ that $L(\varphi)$ is regular (and explicitely construct an automaton for it).

Case $\varphi = Q_a(x)$

- $\varphi = Q_a(x)$. Then $free(\varphi) = x$, and the interpretations of φ are encoded as words over $\Sigma \times \{0, 1\}$. The language $L(\varphi)$ is given by

$$L(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ b_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i \in \{0, 1\} \text{ for every } i \in \{1, \ldots, k\}, \text{ and} \\ b_i = 1 \text{ for exactly one index } i \in \{1, \ldots, k\} \text{ such that } a_i = a \end{array} \right\}$$

and is recognized by

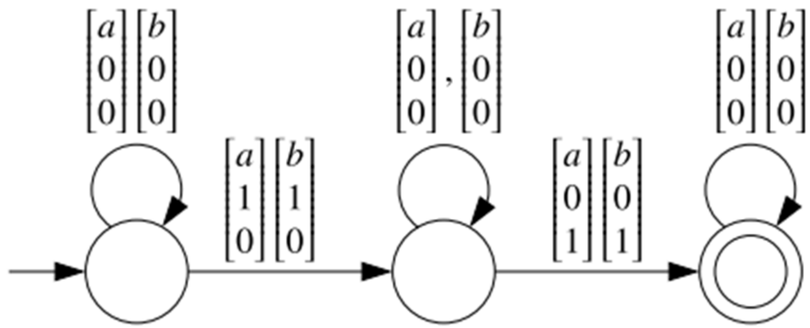


Case $\varphi = x < y$

- $\varphi = x < y$. Then $\mathit{free}(\varphi) = \{x, y\}$, and the interpretations of ϕ are encoded as words over $\Sigma \times \{0,1\}^2$. The language $L(\varphi)$ is given by

$$L(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i, c_i \in \{0,1\} \text{ for every } i \in \{1, \ldots, k\}, \\ b_i = 1 \text{ for exactly one index } i \in \{1, \ldots, k\}, \\ c_j = 1 \text{ for exactly one index } j \in \{1, \ldots, k\}, \text{ and} \\ i < j \end{array} \right\}$$

and is recognized by

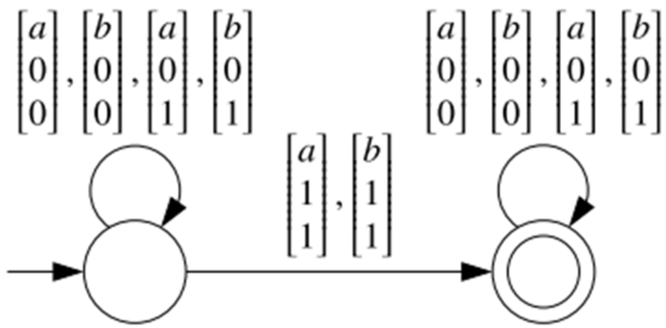


Case $\varphi = x \in X$

- $\varphi = x \in X$. Then $free(\varphi) = \{x, X\}$, and interpretations are encoded as words over $\Sigma \times \{0,1\}^2$. The language $L(\varphi)$ is given by

$$L(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i, c_i \in \{0,1\} \text{ for every } i \in \{1, \ldots, k\}, \\ b_i = 1 \text{ for exactly one index } i \in \{1, \ldots, k\}, \text{ and} \\ \text{for every } i \in \{1, \ldots, k\}, \text{ if } b_i = 1 \text{ then } c_i = 1 \end{array} \right\}$$

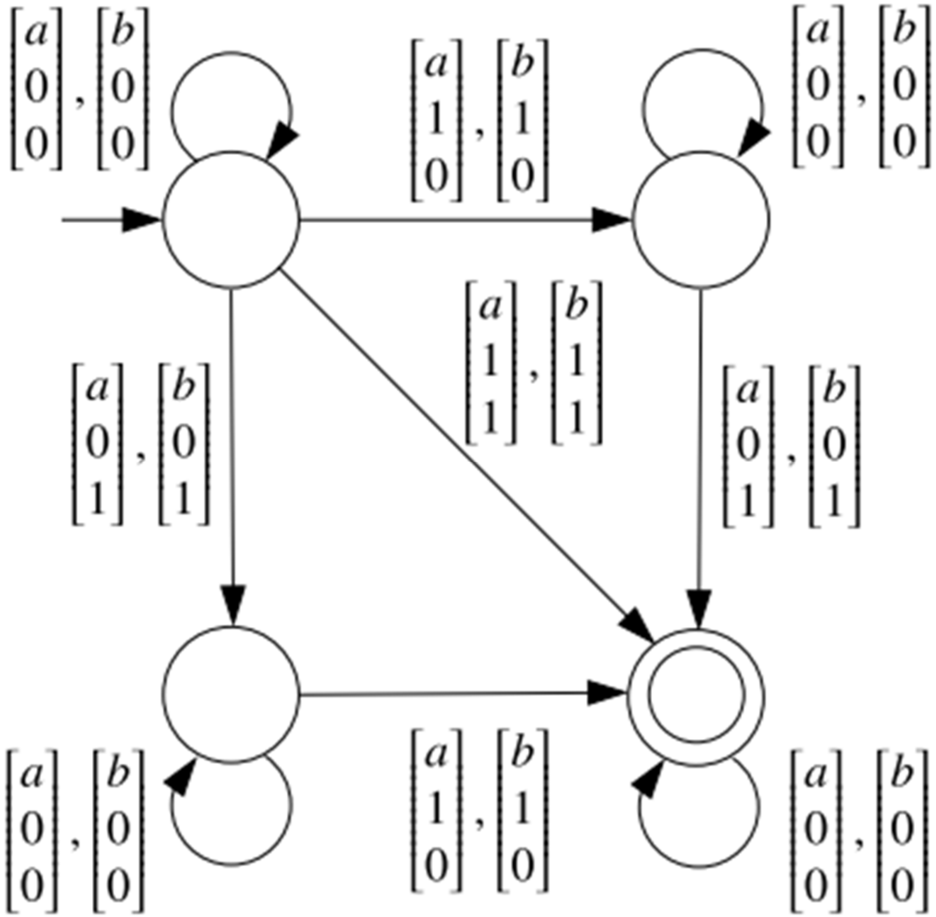
and is recognized by



Case $\varphi = \neg\psi$

- Then $free(\varphi) = free(\psi)$. By i.h. $L(\psi)$ is regular.
- $L(\varphi)$ is equal to $\overline{L(\psi)}$ minus the words that do not encode any implementation („the garbage").
- Equivalently, $L(\varphi)$ is equal to the intersection of $\overline{L(\psi)}$ and the encodings of all interpretations of ψ.
- We show that the set of these encodings is regular.
  - Condition for encoding: Let x be a free first-oder variable of ψ . The projection of an encoding onto x must belong to 0^*10^* (because it represents one position).
  - So we just need an automaton for the words satisfying this condition for every free first-order variable.

Example: $free(\varphi) = \{x, y\}$

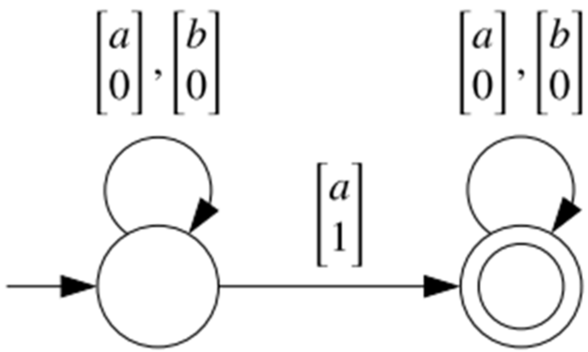


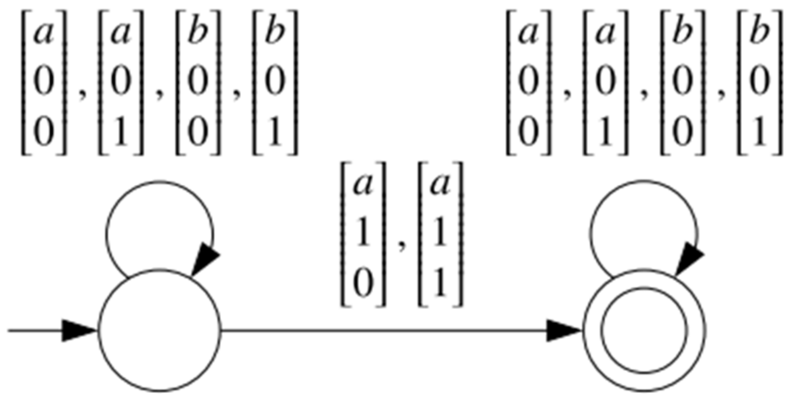
Case $\varphi = \varphi_1 \vee \varphi_2$

- Then $free(\varphi) = free(\varphi_1) \cup free(\varphi_2)$. By i.h. $L(\varphi_1)$ and $L(\varphi_2)$ are regular.
- If $free(\varphi_1) = free(\varphi_2)$ then $L(\varphi) = L(\varphi_1) \cup L(\varphi_2)$ and so $L(\varphi)$ is regular.
- If $free(\varphi_1) \neq free(\varphi_2)$ then we extend $L(\varphi_1)$ to a language L_1 encoding all interpretations of $free(\varphi_1) \cup free(\varphi_2)$ whose projection onto $free(\varphi_1)$ belongs to $L(\varphi_1)$. Similarly we extend $L(\varphi_2)$ to L_2. We have
  - L_1 and L_2 are regular.
  - $L(\varphi) = L_1 \cup L_2$.

Example: $\varphi = Q_a(x) \vee Q\_b(y)$

- L_1 contains the encodings of all interpretations $(w, \{x \mapsto n_1, y \mapsto n_2\})$ such that the encoding of $(w, \{x \mapsto n_1\})$ belongs to $L(Q_a(x))$.
- Automata for $L(Q_a(x))$ and L_1:



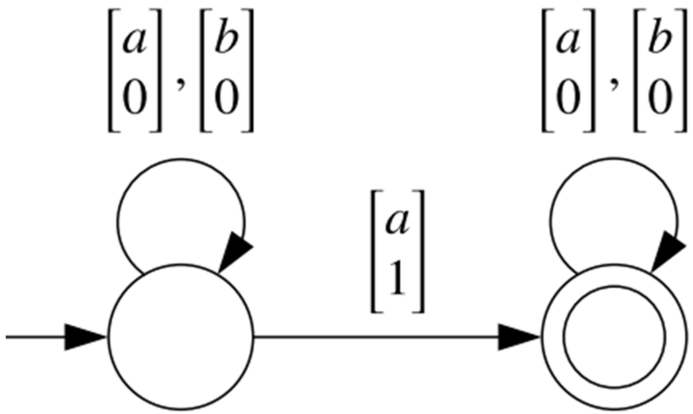


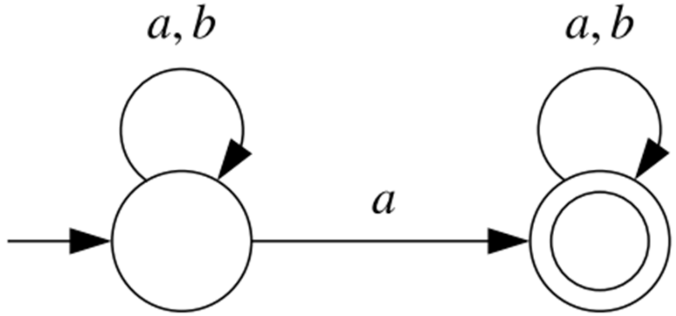
Cases $\varphi = \exists x\ \psi$ and $\varphi = \exists X\ \psi$

- Then $free(\varphi)= free(\psi)\setminus\{x\}$ or $free(\varphi)= free(\psi)\setminus\{X\}$
- By i.h. $L(\psi)$ is regular.
- $L(\varphi)$ is the result of projecting $L(\psi)$ onto the components for $free(\psi)\setminus\{x\}$ or $free(\psi)\setminus\{X\}$.

Example: $\varphi = Q_a(x)$

- Automata for $Q_a(x)$ and $\exists x\, Q_a(x)$





The mega-example

- We compute an automaton for

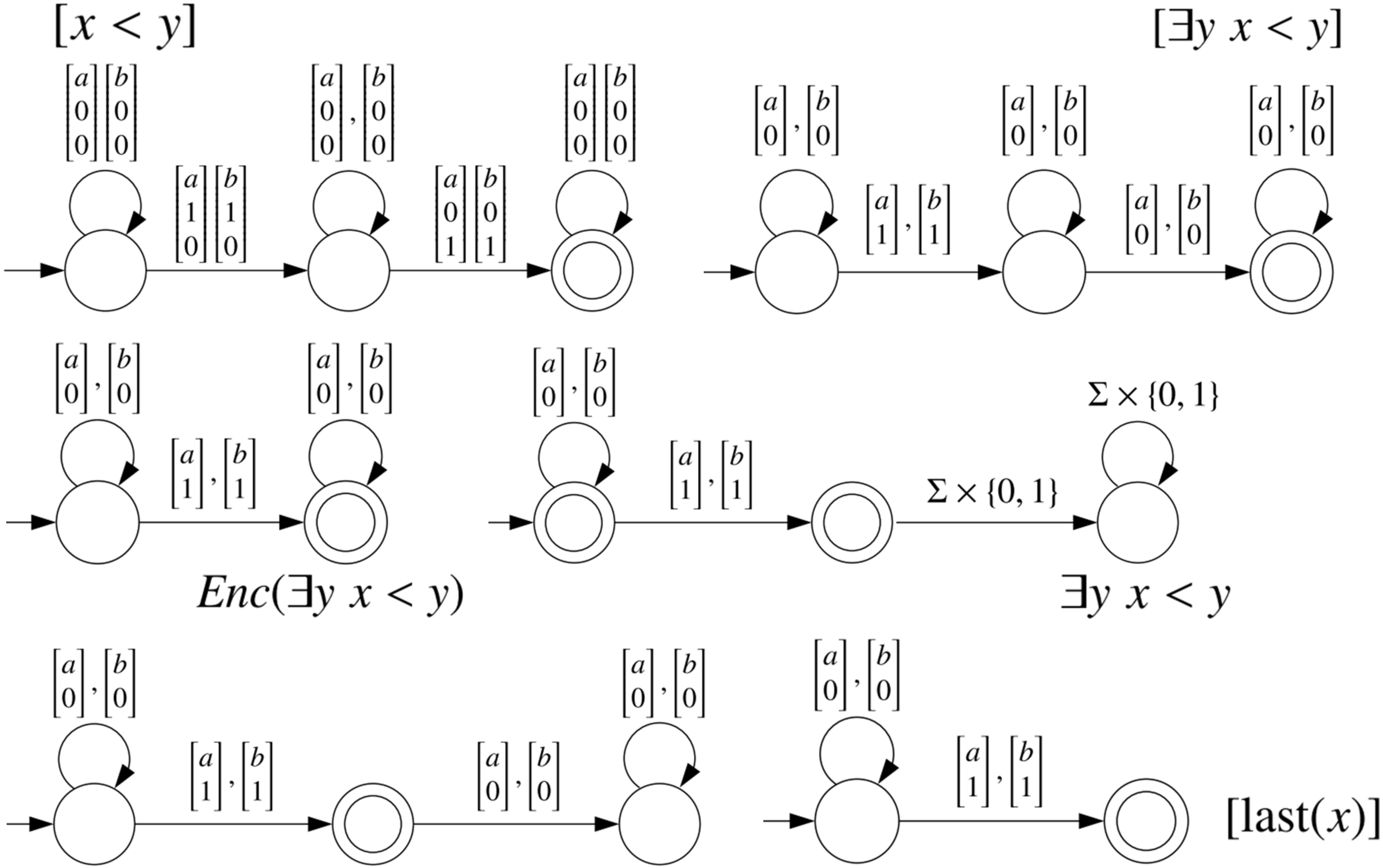
$$\exists x\ (\text{last}(x) \wedge Q_b(x))\ \wedge\ \forall x\ (\neg\text{last}(x) \rightarrow Q_a(x))$$

- First we rewrite φ into

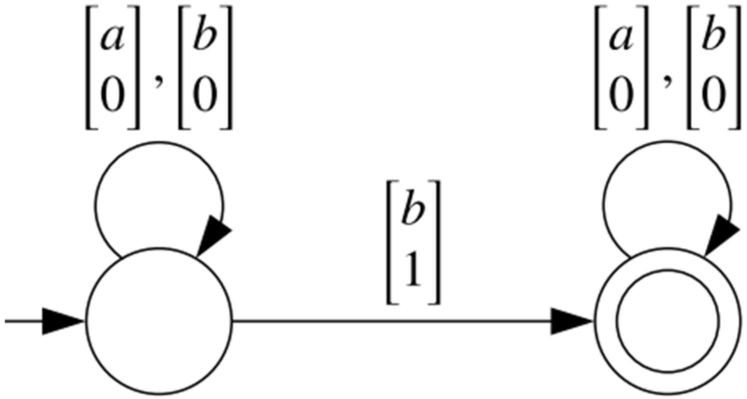
$$\exists x\ (\text{last}(x) \wedge Q_b(x))\ \wedge\ \neg\exists x\ (\neg\text{last}(x) \wedge \neg Q_a(x))$$

- In the next slides we
  1. compute a DFA for $last(x)$
  2. compute DFAs for $\exists x\ (last(x) \wedge Q_b(x))$ and $\neg\exists x\ (\neg last(x) \wedge \neg Q_a(x))$
  3. compute a DFA for the complete formula.
- We denote the DFA for a formula ψ by $[\psi]$.

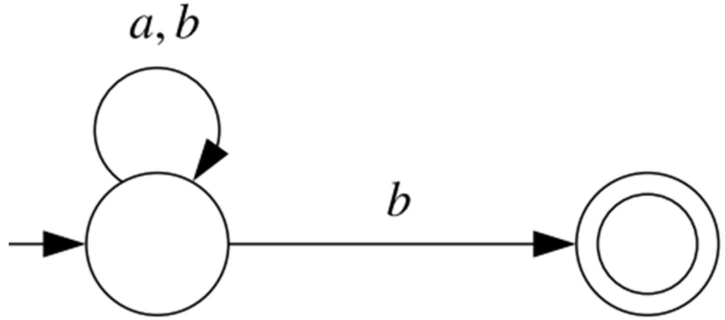
$[last(x)]$



$$[\exists x \left(last(x) \wedge Q_b(x) \right)]$$

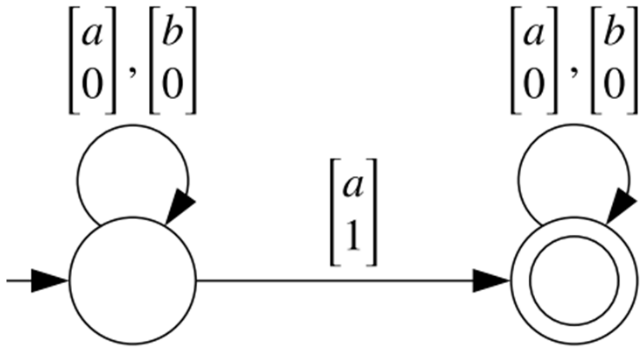


$[Q_b(x)]$

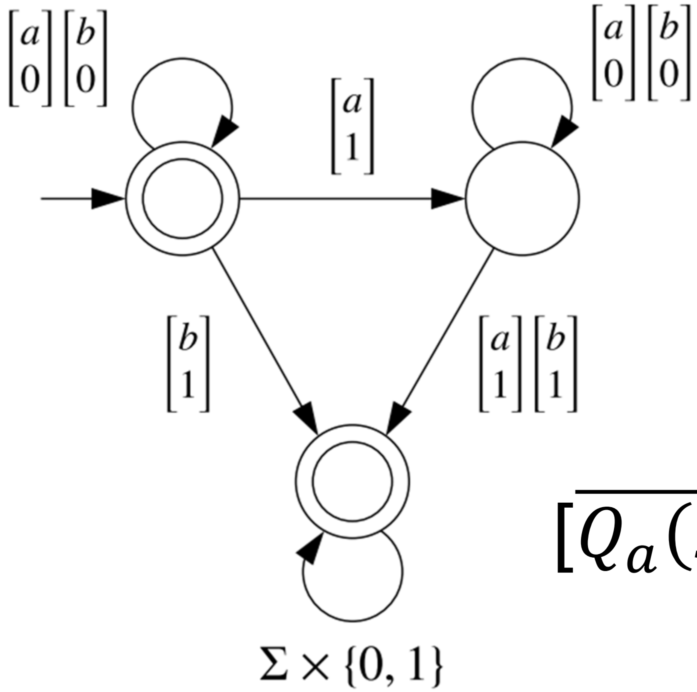


$[\exists x \ (\mathrm{last}(x) \wedge Q_b(x))]$

$[\neg Q_a(x)]$

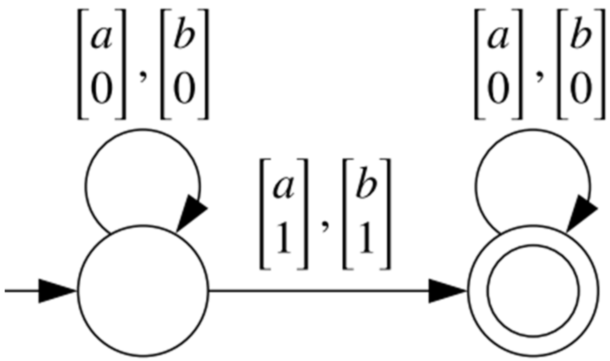


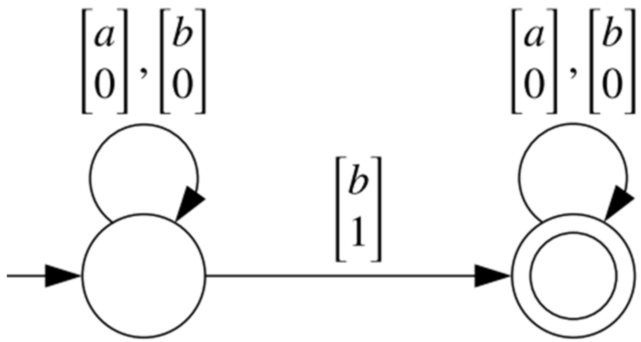
$[Q_a(x)]$



$[\overline{Q_a(x)}]$

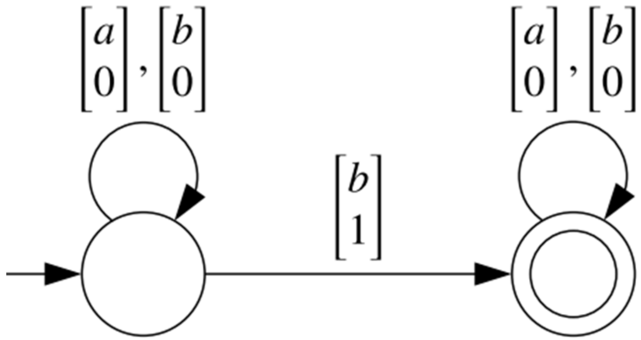
$Enc(Q_a(x))$



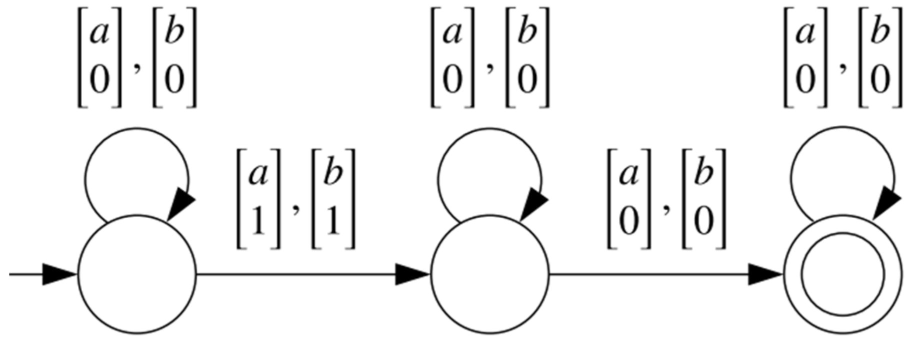


$[\neg Q_a(x)]$

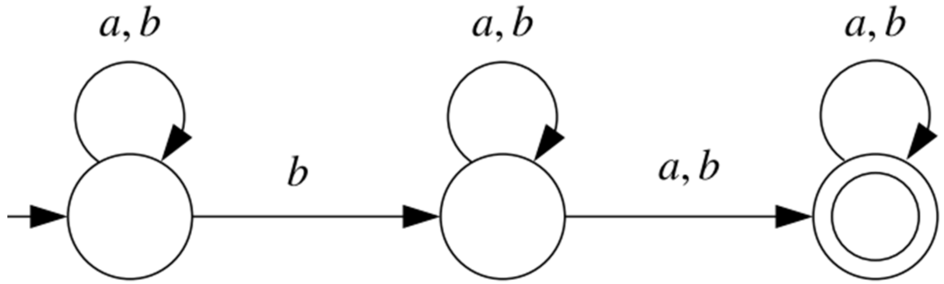
$$[\neg \exists x\, (\neg last(x) \wedge \neg Q_a(x))]$$



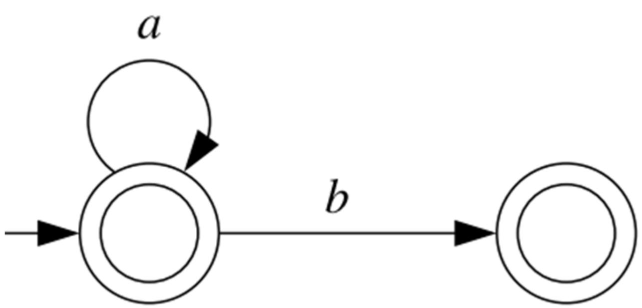
$[\neg Q_a(x)]$



$[\neg last(x)]$

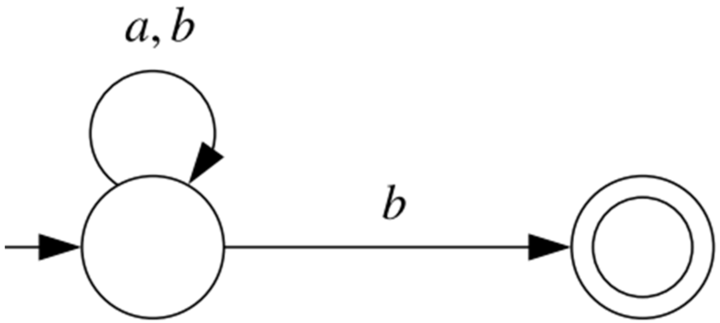


$[\exists x\, (\neg last(x) \wedge \neg Q_a(x))]$

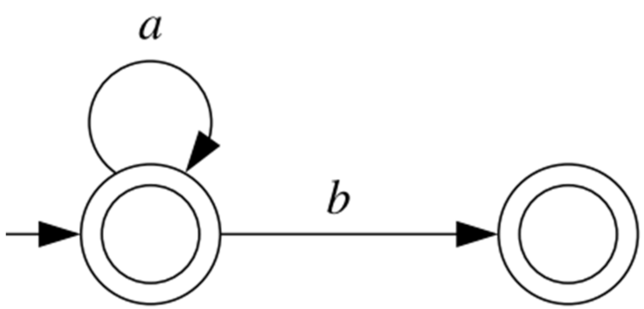


$[\neg \exists x\, (\neg last(x) \wedge \neg Q_a(x))]$

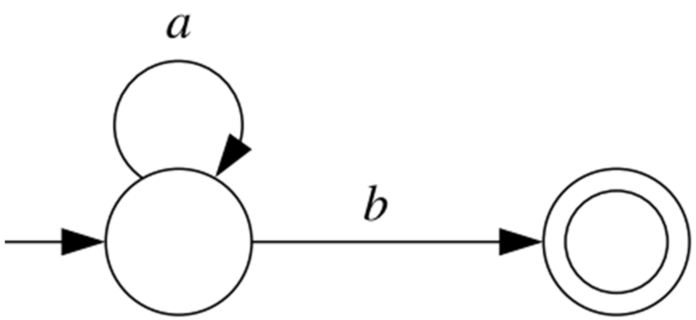
$$[\exists x \left(last(x) \wedge Q_b(x)\right) \wedge \neg\exists x \left(\neg last(x) \wedge \neg Q_a(x)\right)]$$



$[\exists x\ (\mathrm{last}(x) \wedge Q_b(x))]$



$[\neg\exists x\ (\neg last(x) \wedge \neg Q_a(x))]$



$[\exists x\ (\mathrm{last}(x) \wedge Q_b(x)) \ \wedge\ \neg\exists x\ (\neg\mathrm{last}(x) \wedge \neg Q_a(x))]$