Automata and Formal Languages — Sample Solution 9

Due 11.12.2015

Solution 9.1

(a) $\exists x \text{ first}(x)$

(b) $\exists x\ (\text{first}(x) \wedge Q_a(x))$

(c) $\forall x\ (Q_a(x) \to \exists y\ (x < y \wedge Q_b(y))) \wedge \forall x\ (Q_b(x) \to \exists y\ (x = y + 1 \wedge Q_a(y)))$

(d) $\neg\exists x \exists y\ (x < y \wedge Q_b(x) \wedge Q_a(y))$

(e) $\forall x(Q_a(x) \to \forall y(y = x + n \to Q_a(y))) \wedge \forall x(Q_b(x) \to \forall y(y = x + n \to Q_b(y)))$

(f) Define

$$y = x + 2^n \quad := \quad \exists z \exists v \exists u\ \big(((v = z \wedge u = x) \vee (v = y \wedge u = z)) \to v = u + 2^{n-1}\big)\ .$$

L'_n can be defined similarly to (e), by replacing $y = x + n$ with $y = x + 2^n$. Note that the size of $y = x + 2^n$ is $O(n)$, and therefore the size of L'_n is also $O(n)$.

Solution 9.2

(a) Define

$$\text{Odd}(X) \quad := \quad \forall x\ \ (x \in X \leftrightarrow (\text{first}(x) \vee \exists y\ (x = y + 2 \wedge y \in X)))\ .$$

We have the formula: $\exists X\ (\text{Odd}(X) \wedge (\forall x \in X\ Q_a(x)) \wedge \forall x\ (\text{last}(x) \to (x \in X \vee \exists y \in X\ (x = y + 1))))$.

(b) Define

$$\text{OddLength} \quad := \quad \exists X\ (\text{Odd}(X) \wedge \forall x\ (\text{last}(x) \to x \in X))\ .$$

We have the formula: $\text{OddLength} \wedge \forall x\ Q_a(x)$.

(c) Define the following abbreviations:

$$\begin{array}{lll}
\text{First}(x, X) & := & x \in X \wedge \forall y\ (y < x \to y \notin X) \\
\text{Last}(x, X) & := & x \in X \wedge \forall y\ (x < y \to y \notin X) \\
\text{Next}(x, y, X) & := & x \in X \wedge y \in X \wedge \neg\exists z \in X\ (x < z \wedge z < y) \\
\text{Odd}(Y, X) & := & \forall x\ \ (x \in Y \leftrightarrow (\text{First}(x, X) \vee \exists y \exists u\ (y \in Y \wedge \text{Next}(y, u, X) \wedge \text{Next}(u, x, X)))) \\
\text{Odd_card}(X) & := & \exists Y\ \ (\text{Odd}(Y, X) \wedge \forall x\ (\text{Last}(x, X) \to x \in Y))\ .
\end{array}$$

Intuitively, $\text{First}(x, X)/\text{Last}(x, X)$ is true when x is the first/last position in X. $\text{Next}(x, y, X)$ expresses that y is the successor of x in X. $\text{Odd}(Y, X)$ means that Y contains all odds positions of X, and $\text{Odd\_card}(X)$ means that the cardinality of X is an odd number.

We have the formula: $\forall X\ ((\forall x \in X\ Q_a(x)) \to \text{Odd\_card}(X))$.

(d) $\text{OddLength} \wedge \forall X\ ((\forall x \in X\ Q_a(x)) \to \text{Odd\_card}(X))$.

(e) $\forall X\ (\text{Cons}(X) \wedge \forall x \in X\ (Q_b(x) \leftrightarrow (\text{first}(x, X) \vee \text{last}(x, X))) \to \text{Odd\_card}(X))$

Solution 9.3

Given $\psi \in \mathit{Pure\_MSO}(\Sigma)$, we can construct an equivalent $\phi \in \mathit{MOS}(\Sigma)$ by replacing each subformula of the form

$$\begin{array}{lll} X \subseteq Q_a & \text{by} & \forall x \in X \; Q_a(x) \\ X < Y & \text{by} & \forall x \in X \; \forall y \in Y \; x < y \\ X \subseteq Y & \text{by} & \forall x \in X \; x \in Y \; . \end{array}$$

Given $\phi \in \mathit{MSO}(\Sigma)$, we can construct an equivalent $\psi \in \mathit{Pure\_MOS}(\Sigma)$ by replacing each subformula of the form

$$\begin{array}{lll} Q_a(x) & \text{by} & X \subseteq Q_a \\ x < y & \text{by} & X < Y \\ x \in Y & \text{by} & X \subseteq Y \\ \exists x \; \psi' & \text{by} & \exists X \; (\mathrm{sing}(X) \wedge \psi'[x/X]) \; , \end{array}$$

where X is a fresh second-order variable that does not appear in ϕ, the formula $\psi'[x/X]$ is the result of substituting X for x in ψ', and

$$\mathrm{sing}(X) \quad := \quad \exists x \in X \; \forall y \in X \; (x = y) \; .$$