Automata and Formal Languages — Endterm

Answer all questions. Question 1 requires only short answers that can be written directly on this sheet. All other questions should be answered in a separate booklet. A total of 40 points is available.

Question 1

(a) Prove (via a proof) or disprove (via a counterexample) the following LTL equivalence.

\[(Gp) U (Gq) \equiv G(p U q)\]

Answer: \[1\] point

(b) Let \(\Sigma = \{a, b\}\). Give formulae of MSO(\(\Sigma\)) for each of the following two languages. You may use the following macros from the lecture notes:

\[
\begin{align*}
\text{first}(x) & := \neg\exists y \ y < x \\
y = x + 1 & := \ x < y \land \neg\exists z \ (x < z \land z < y) \\
y = x + 2 & := \exists z \ (z = x + 1 \land y = z + 1) \\
X \subseteq Y & := \forall x \ (x \in X \rightarrow x \in Y) \\
\text{Odd}(X) & := \forall x \ (x \in X \leftrightarrow (\text{first}(x) \lor \exists z \ (x = z + 2 \land z \in X)))
\end{align*}
\]

(i) \(ab^*\)

Answer:

(ii) \((ab + ba)^*\)

Answer: \[2\] points
(c) Consider the following Müller automaton with $\mathcal{F} = \{\{q_0, q_1\}, \{q_1\}\}$:

```
   q0 ---a--> q1
       |       b
       v       
   q1 ----a--> q0
```

(i) Describe the language recognized by the automaton.

Answer: 

(ii) Construct an equivalent deterministic Büchi automaton.

Answer: 

2 points

(d) Consider the following Büchi automaton:

```
   q ----a--> r
       |       b
       v       
   r ----b--> q

   t ----a--> s
```

Which lassos can be reported by the NestedDFS algorithm?

Answer: 

2 points

(e) Draw a deterministic co-Büchi automaton recognising the language $(a + b)^*(aa + bb)\omega$.

Answer: 

2 points
(f) Fill in the gaps in the following algorithm (in lines 2, 7 and 11) so that it meets its specification.

Input: DFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1), A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
Output: DFA $A = (Q, \Sigma, \delta, q_0, F)$ with $L(A) = (L(A_1) \cap L(A_2)) \cup \overline{L(A_2)}$

1. $Q, \delta, F \leftarrow \emptyset$
2. $q_0 \leftarrow \emptyset$
3. $W \leftarrow \{q_0\}$
4. while $W \neq \emptyset$ do
5.   pick $[q_1, q_2]$ from $W$
6.   add $[q_1, q_2]$ to $Q$
7.   if $[q_1', q_2'] \not\in Q$ then $[q_1', q_2']$ to $W$
8.   for all $a \in \Sigma$ do
9.     $q_1' \leftarrow \delta_1(q_1, a); q_2' \leftarrow \delta_2(q_2, a)$
10.    if $[q_1', q_2'] \not\in Q$ then add $[q_1', q_2']$ to $W$
11.   add $\text{[1 point]}$ to $\delta$ [1 point]

(g) Draw a Büchi automaton recognising the infinite sequences $\sigma$ of subsets of $\{p, q\}$ such that

$\sigma \models G(F(p U q))$

(You do not need to use the conversion algorithm from the lectures).

Answer: $\text{[2 points]}$

[Total: 12 points]
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Question 2

Given two languages $L_1$ and $L_2$ over an alphabet $\Sigma$, define the language $L_{12}$ as follows:

\[ w \in L_{12} \iff \text{there is a word } u_1u_2 \in L_1 \text{ and a word } v \in L_2 \text{ such that } w = u_1v. \]

Show that if $L_1$ and $L_2$ are regular, then $L_{12}$ is regular.

[Total: 4 points]

Question 3

(a) Draw a DFA recognising $(a + bb)^*$. 
(b) Given a language $L \subseteq \Sigma^*$ and $w \in \Sigma^*$, give a formal definition of the residual $L^w$. 
(c) Give regular expressions for every residual of the language $(a + bb)^*$. 
(d) Prove that the language $L' := \{ a^{2^n} \mid n \in \mathbb{N}_0 \}$ has infinitely many residuals. Is $L'$ regular? Justify your answer.

[Total: 6 points]

Question 4

For this question, use the LSBF encoding of the natural numbers $\mathbb{N}_0$.

(i) Given a binary relation $R \subseteq \mathbb{N}_0 \times \mathbb{N}_0$, what does it mean to say that $R$ is regular? 
(ii) Show that the relation $R_1 := \{ (n, 2n) \mid n \in \mathbb{N}_0 \}$ is regular. 
(iii) Show that the relation $R_2 := \{ (n, 1 + n) \mid n \in \mathbb{N}_0 \}$ is regular. 
(iv) You may assume that given two regular relations $P_1$ and $P_2$, their composition $P_1 \circ P_2 := \{ (n, m) \mid (n, k) \in P_1 \text{ and } (k, m) \in P_2 \text{ for some } k \in \mathbb{N}_0 \}$ is also regular. Use this fact together with the facts that $R_1$ and $R_2$ from parts (ii) and (iii) are regular to show that the following relation is regular $R := \{ (n, 4n + 3) \mid n \in \mathbb{N}_0 \}$

[Total: 6 points]
Question 5

Construct a LazyDFA recognising words over the alphabet \{a, b, c\} that contain the pattern \textit{abacab}. \hfill [Total: 3 points]

Question 6

Consider the following program with two parallel processes. The domain of \(x\) is \{0, 1\}, and the initial value is 0.

Process 1:
\[
\textbf{while true do} \\
1\quad \textbf{if } \ x = 0 \quad \textbf{then} \\
2\quad \ x \leftarrow 1
\]

Process 2:
\[
\textbf{while true do} \\
1\quad \ x \leftarrow 0
\]

(a) Model the program by constructing a network of three automata (one for each process and one for the variable \(x\)) and then drawing their asynchronous product.

(b) Consider the set of atomic propositions \(\mathcal{AP} = \{x = 0, x = 1\}\). Give an LTL formula for the following property: \(x = 0\) holds infinitely often. Construct a Büchi automaton for the negation of this property.

(c) Does the program satisfy the property in (b)? Why? \hfill [Total: 5 points]

Question 7

⋆

(a) Consider the language \(L := \{0^i \mid i \text{ is prime}\}\) (which is not regular). Prove that \(L^* = \epsilon + 000^*\).

(b) Now let \(L\) be an \textit{arbitrary} language over the single-letter alphabet \{0\}. (\(L\) is not necessarily regular). Prove that \(L^*\) is regular. \hfill [Total: 4 points]

End of Examination