Technische Universität München **Winter term 2015/16**
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Automata and Formal Languages — Homework 14

Due **Friday** 29th December 2016 (TA: Christopher Broadbent)

Exercise 14.1
(Schwoon). Which of the following formulas of LTL are tautologies? (A formula is a tautology if all computations satisfy it.) If the formula is not a tautology, give a computation that does not satisfy it.

- $\mathbf{G}p \rightarrow \mathbf{F}p$
- $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$
- $\mathbf{F}(p \wedge q) \leftrightarrow (\mathbf{F}p \wedge \mathbf{F}q)$
- $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$
- $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$
- $(\mathbf{FG}p \rightarrow \mathbf{GF}q) \leftrightarrow \mathbf{G}(p\ \mathbf{U}\ (\neg p \vee q))$
- $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$.

Exercise 14.2
Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give LTL formulas defining the following languages:

$$\{p,q\}\ \emptyset\ \Sigma^\omega \qquad \Sigma^*\ \{q\}^\omega$$
$$\Sigma^*\ (\{p\} + \{p,q\})\ \Sigma^*\ \{q\}\ \Sigma^\omega \qquad \{p\}^*\ \{q\}^*\ \emptyset^\omega$$

Exercise 14.3
Let $AP = \{p, q\}$. Give NBAs accepting the languages defined by the LTL formulas $\mathbf{XG}\neg p$, $(\mathbf{GF}p) \Rightarrow (\mathbf{F}q)$, and $p \wedge \neg\mathbf{XF}p$.

Exercise 14.4
Prove $\mathbf{FG}p \equiv \mathbf{VFG}p$ and $\mathbf{GF}p \equiv \mathbf{VGF}p$ for *every* sequence $\mathbf{V} \in \{\mathbf{F}, \mathbf{G}\}^*$ of the temporal operators $\mathbf{F}$ and $\mathbf{G}$.

Solution 14.1

The formulas in red are tautologies.

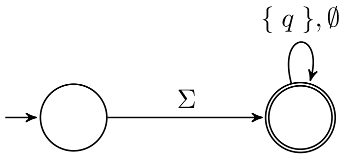
- $\mathbf{G}p \to \mathbf{F}p$.
  Follows immediately from the definitions of $\mathbf{F}$ and $\mathbf{G}$.
- $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$.
  The left-hand-side states that any point of the computation satisfying p also satisfies q. Therefore, if every point satisfies p, tehn every point satisfies q.
- $\mathbf{F}(p \wedge q) \leftrightarrow (\mathbf{F}p \wedge \mathbf{F}q)$.
  The computation $\{p\}\,\{q\}\,\emptyset^\omega$ satisfies $\mathbf{F}p \wedge \mathbf{F}q$ but not $\mathbf{F}(p \wedge q)$.
- $\neg\mathbf{F}p \to \mathbf{F}\neg\mathbf{F}p$.
  The formula $\phi \to \mathbf{F}\phi$ is clearly a tautology for every formula ϕ. Take $\phi = \neg\mathbf{F}p$.
- $(\mathbf{G}p \to \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$.
  The left-hand-side is equivalent to $\mathbf{F}\neg p \vee \mathbf{F}q \equiv \mathbf{F}(\neg p \vee q)$. If the right-hand-side holds, then some point of the computation satisfies $\neg p \vee q$, and so the left-hand-side holds. If the left-hand-side holds, then there exists a first point at which $\neg p \vee q$ holds, and. since it is the first, all points before it satisfy $p \wedge \neg q$, and so in particular they all satisfy p. So the right-hand-side holds as well.
- $(\mathbf{FG}p \to \mathbf{GF}q) \leftrightarrow \mathbf{G}(p\ \mathbf{U}\ (\neg p \vee q))$.
  The left-hand-side is equivalent to $\mathbf{GF}\neg p \vee \mathbf{GF}p$, which is equivalent to $\mathbf{GF}(\neg p \vee q)$ (recall that $\mathbf{GF}$ means "infinitely often"). If the right-hand-side holds, then every point satisfies $\mathbf{F}\neg p \vee \mathbf{GF}p$, and so for every point some future point satisfies $\neg p \vee q$, which implies that the left-hand-side holds. If the left-hand-side holds, then infinitely many points satisfy $\neg p \vee q$, and all others satisfy p. Therefore, every point satisfies $\mathbf{G}(p\ \mathbf{U}\ (\neg p \vee q))$.
- $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$.
  We have $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p) \equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p \equiv \mathbf{F}\neg p \vee \mathbf{G}p$, which is clearly a tautology.

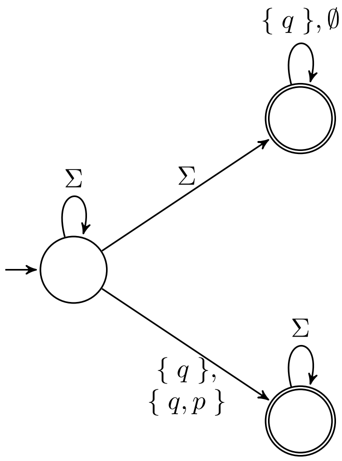
Solution 14.2

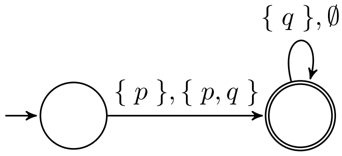
$(p \wedge q) \wedge \mathbf{X}(\neg p \wedge \neg q)$	$\mathbf{F}(\neg p \wedge q)$
$\mathbf{F}(p \wedge \mathbf{XF}(\neg q \wedge q))$	$(p \wedge \neg q)\ \mathbf{U}\ ((\neg p \wedge q)\ \mathbf{U}\ \mathbf{G}(\neg p \wedge \neg q))$

Solution 14.3

Taking $\Sigma := 2^{AP}$.







Solution 14.4

Given two formulas ϕ, ψ of LTL, we denote by $\phi \models \psi$ that every computation satisfying ϕ satisfies ψ. Clearly, we have $\phi \equiv \sigma$ iff $\phi \models \psi$ and $\psi \models \phi$. We prove several little lemmas.

(1) For every formula ϕ: $\mathbf{FF}\phi \equiv \mathbf{F}\phi$ and $\mathbf{GG}\phi \equiv \mathbf{G}\phi$.
Follows immediately from the definitions.

(2) For every formula ϕ: $\mathbf{G}\phi \models \phi$ and $\phi \models \mathbf{F}\phi$.
Follows immediately from the definitions.

(3) For every formula ϕ: $\mathbf{FG}\phi \equiv \mathbf{GFG}\phi$.
$\mathbf{GFG}\phi \models \mathbf{FG}\phi$ by (2). $\mathbf{FG}\phi \models \mathbf{GFG}\phi$ because if there is point at a computation such that from this point onwards ϕ always holds, then the same is true of every suffix of the computation.

(4) For every formula ϕ: $\mathbf{GF}\phi \equiv \mathbf{FGF}\phi$.
$\mathbf{GF}\phi \models \mathbf{FGF}\phi$ by (2). $\mathbf{FGF}\phi \models \mathbf{GF}\phi$ because if p holds at infinitely many points of some suffix of a computation, then it also holds at infinitely many points of the computation.

We prove $\mathbf{FG}p \equiv \mathbf{VFG}p$ by induction on the length of $\mathbf{V}$. If $\mathbf{V} = \epsilon$, we are done. If $\mathbf{V} = \mathbf{YF}$, then we have $\mathbf{VFG}p \equiv \mathbf{YFG}p$ by (1), and if $\mathbf{V} = \mathbf{YG}$, then we have the same equivalence by (3). By induction hypothesis we get $\mathbf{YFG}p \equiv \mathbf{FG}p$. The other equivalence is proved similarly.