Technische Universität München **Winter term 2015/16**
I7
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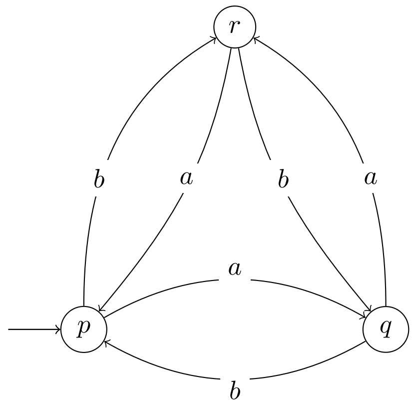
Automata and Formal Languages — Homework 12

Due **Friday** 15th December 2016 (TA: Christopher Broadbent)

Exercise 12.1

Consider the following deterministic Müller automaton A over the alphabet $\{ a, b \}$ with acceptance condition

$$\{ \{ p, q \}, \{ q, r \}, \{ r, p \} \}$$



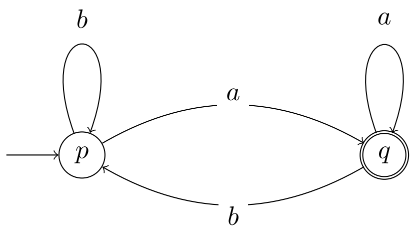
(i) Write down an ω-regular expression for the language recognised by A.

(ii) Write down a new Müller acceptance condition that would result in an automaton recognising $\overline{L(A)}$.

(iii) Describe a procedure for directly complementing deterministic Müller automata (without going through Büchi automata).

(iv) Describe a procedure that takes two deterministic Müller automata A_1 and A_2 as input and returns a deterministic Müller automaton B such that $L(B) = L(A_1) \cup L(A_2)$.

Exercise 12.2



(i) Viewing the automaton above as a deterministic *finite automaton*, write down a regular expression for the finite language that it recognises.

(ii) Viewing the automaton above as a deterministic *Büchi automaton*, write down an ω-regular expression for the ω-language that it recognises.

(iii) The *limit* of a language $L \subseteq \Sigma^*$, denoted by $lim(L)$, is the ω-language defined as follows: $w \in lim(L)$ iff infinitely many prefixes of w are words of L. For example, the limit of $(ab)^*$ is $\{(ab)^\omega\}$.

Prove that an ω-language is recognisable by a deterministic Büchi automaton iff it is the limit of a regular language.

(iv) Exhibit a non-regular language whose limit is ω-regular.

(v) Exhibit a non-regular language whose limit is *not* ω-regular.

(vi) Exhibit an ω-regular language (recognised by a *non-deterministic* Büchi automaton) that is *not* the limit of any regular language.

Exercise 12.3

The *parity acceptance condition* for ω-automata is defined as follows. Every state q of the automaton is assigned a natural number n_q. A run ρ is accepting if the number $\max\{n_s \mid s \in \mathit{inf}(\rho)\}$ is even.

(i) Find a parity automaton accepting the language $L = \{w \in \{a, b\}^\omega \mid w \text{ has exactly two occurrences of } ab\}$.

(ii) Show that each language accepted by a parity automaton is also accepted by a Rabin automaton and vice versa.

Solution 12.1

(i) The automaton recognises the language $(a+b)^*(a+b)^\omega$.

(ii) We can simply complement the Müller acceptance condition to get:

$$\{\ \{\ p,q,r\ \},\{\ p\ \},\{\ q\ \},\{\ r\ \},\emptyset\ \}$$

Every infinite run must contain at least one state that occurs infinitely often (there are only finitely many possible states to choose from), and so it never makes any difference including $\emptyset$ in a Müller acceptance condition. Moreover, in this particular case there are no self-loops in the automaton and so it is not possible for just one state to occur infinitely often. Thus in this particular case, the following would also be a correct answer:

$$\{\ \{\ p,q,r\ \}\ \}$$

(iii) If a *deterministic* Müller automaton has acceptance condition $\Omega \in 2^Q$ (where 2^Q is the power set of the set of states Q), then it can always be complemented by replacing its acceptance condition with $\overline{\Omega} := 2^Q - \Omega$.

(iv) We can reuse the product construction for deterministic finite automata to construct the state space Q , transition relation and initial states of B. For the acceptance condition Ω of B we take:

$$\Omega := \{\ S \subseteq Q \mid \pi_1(Q) \in \Omega_1 \text{ or } \pi_2(Q) \in \Omega_2\ \}$$

where Ω_1 and Ω_2 are the respective acceptance conditions for A_1 and A_2 (and π_1 and π_2 are respectively the first and second projections so that $\pi_i(S) := \{\ q_i \mid (q_1, q_2) \in S\ \}$).

Solution 12.2

(i) $(b^*a)(b^+a\ +\ a)^*$

(ii) $(b^*a)(b^+a\ +\ a)^\omega$

(iii) Let B be a deterministic Büchi automaton recognizing an ω-language L. Look at B as a DFA, and let L' be the regular language recognized by B. We prove $L = \mathit{lim}(L')$. If $w \in \mathit{lim}(L')$, then B (as a DFA) accepts infinitely many prefixes of w. Since B is deterministic, the runs of B on these prefixes are prefixes of the unique infinite run of B (as a DBA) on w. So the infinite run visits accepting states infinitely often, and so $w \in L$. If $w \in L$, then the unique run of B on w (as a DBA) visits accepting states infinitely often, and so infinitely many prefixes of w are accepted by B (as a DFA). So $w \in \mathit{lim}(L')$.

(iv) The irregular language $\{\ a^n b^n \mid n \in \omega\ \}$ has limit $\emptyset$, which is trivially ω-regular. In order to pick a less degenerate example, the irregular language $\{\ a^{2^n} \mid n \in \omega\ \}$ has as its limit a^ω, which is clearly ω-regular.

(v) The irregular language $\{\ a^n b^n c^i \mid n \in \omega, i \in \omega\ \}$ has limit $\{\ a^n b^n c^\omega \mid n \in \omega\ \}$, which is not ω-regular.

(vi) The ω-regular language $L_\omega := (a+b)^* a^\omega$ is not the limit of any regular language of finite words. (It follows that $(a+b)^* a^\omega$ cannot be recognised by a deterministic Büchi automaton, although it can be recognised by a non-deterministic Büchi automaton).

In order to prove this, suppose for contradiction that there does exist some language L of finite words such that $L_\omega = \mathit{lim}(L)$. (In fact L need not even be regular, so we prove the required result also for irregular finite languages).

Let us write $w_1 \sqsubseteq w_2$ to mean that w_1 is a prefix of w_2.

We now construct an ω-sequence of ω-words $(u_i)_{i\in\omega}$ and an ω-sequence of finite words $(v_i)_{i\in\omega}$. The words u_i will all belong to L_ω, and the words v_i will all belong to L. Moreover it will be the case that for each $i \in \omega$, $v_i b \sqsubseteq v_{i+1}$.

To start with we take

$$u_0 := ba^\omega \qquad \text{and } b \sqsubseteq v_0 \sqsubseteq u_0$$

where v_0 may be any fixed such word in L. Note that such a v_0 must exist, since we are assuming that u_0 has inifinitely many prefixes in L.

We then take

$$u_{i+1} := v_i b a^\omega$$

Since each v_i belongs to $(a+b)^*$ it must be the case that $u_{i+1} \in L_\omega$. Since we assume that u_{i+1} has infinitely many prefixes in L, it must be the case that it has one, to which we set v_{i+1}, of length greater than $|v_0\, v_1\ \cdots\ v_i b|$ so that

$$v_i b \sqsubseteq\ v_{i+1}\ \sqsubseteq\ u_{i+1}$$

Thus we have an ω sequence of words in L such that

$$v_0\, b \sqsubseteq v_1 \quad v_1\, b \sqsubseteq v_2 \quad \cdots \quad v_i\, b \sqsubseteq v_{i+1} \quad \cdots$$

$v_0\, b, \quad v_1\, b, \cdots$ give infinitely many prefixes of some word in $((a+b)^*b)^\omega$, which cannot belong to $(a+b)^*a^\omega$, giving the required contradiction.