Operations on relations: Implementation on NFAs

Projection_1(R) : returns the set $\pi_1(R) = \{x \mid \exists y\ (x, y) \in R\}$.
Projection_2(R) : returns the set $\pi_2(R) = \{y \mid \exists y\ (x, y) \in R\}$.
Join(R_1, R_2) : returns $R_1 \circ R_2 = \{(x, z) \mid \exists y \in X\ (x, y) \in R_1 \wedge (y, z) \in R_2\}$

Post(Y, R) : returns $post_R(Y) = \{x \in X \mid \exists y \in Y : (y, x) \in R\}$.
Pre(Y, R) : returns $pre_R(Y) = \{x \in X \mid \exists y \in Y' : (x, y) \in R\}$.

Encoding objects

- So far we have assumed for convenience:
  a) every word encodes one object.
  b) every object is encoded by exactly one word.

  We now analyze this in more detail.

- Example: objects → natural number encoding → *lsbf*
  lsbf(5) = 101 *lsbf*(0) = ε.
  Satisfies b), but not a).

- We argue that a) can be easily weakened to:
  a') the set of words encoding objects is a regular language.

- The *lsbf* encoding satisfies a'):
  set of encodings → $\{\varepsilon\} \cup \{w \in \Sigma^* \mid w \text{ ends with } 1\}$

Encoding pairs

- Extending the implementations to relations requires to encode pairs of objects.

- How should we encode a pair (n_1, n_2) of natural numbers?

- Consider the pair (n_1, n_2) .
- Assume n_1, n_2 encoded by w_1, w_2 in *lsbf* encoding
- Which should be the encoding of (n_1, n_2) ?
  - Cannot be $w_1 w_2$.
    Then same word encodes many pairs, violates b).

- First attempt: use a separator symbol &, and encode (n_1, n_2) by $w_1 \& w_2$.
  - Problem: not even the identity relation gives a regular language!

- Second attempt: encode (n_1, n_2) as a word over $\{0,1\} \times \{0,1\}$ (intuitively, the automaton reads w_1 and w_2 simultaneously).
  - Problem: what if w_1 and w_2 have different length?
  - Solution: fill the shortest one with $0s$.
  - Satisfies b) and a'), but not (a):
    - The number k is encoded by all the words of $s_k 0^*$, where s_k is the *lsbf* encoding of k.
  - We call 0 the padding symbol or padding letter.

- So we assume:
  - The alphabet contains a padding letter #, different or not from the letters used to encode an object.
  - Each object x has a minimal encoding s_x .
  - The encodings of x are all the words of $s_x\#^*$.
  - A pair (x, y) of objects has a minimal encoding $s_{(x,y)}$.

  - The encodings of (x, y) are all the words of $s_{(x,y)}\#^*$.

- Question: if objects (pairs of objects) are encoded by multiple words, which is the set of objects (pairs) recognized by a DFA or NFA?

  (We can no longer say: an object is recognized if its encoding is accepted by the DFA or NFA!)

- Question: because of the new definition of "set of objects recognized by an automaton", do we have to change the implementation of the set operations?

Definition 5.2 *Assume an encoding of X over Σ^* has been fixed. Let A be an NFA.*

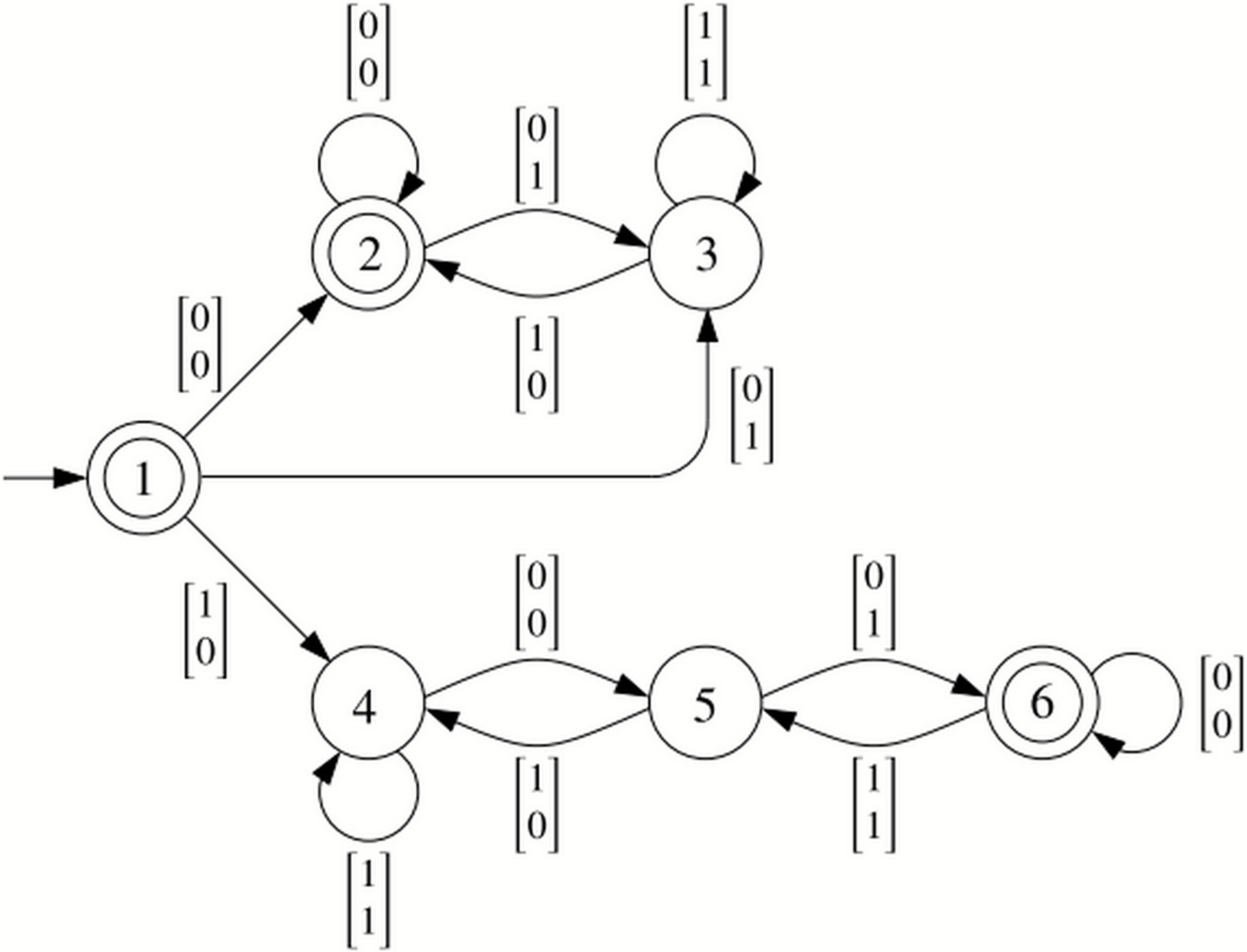
- *A accepts $x \in X$ if it accepts all encodings of x.*
- *A rejects $x \in X$ if it accepts no encoding of x.*
- *A recognizes a set $Y \subseteq X$ if*

$$\mathcal{L}(A) = \{w \in \Sigma^* \mid w \text{ encodes some element of } Y\} .$$

A subset $Y \subseteq X$ is regular *(with respect to the fixed encoding) if it is recognized by some NFA.*

Notice that with this definition a NFA may neither accept nor reject a given x. In this case the NFA does not recognize any subset of X.

Transducers



Definition 5.3 *A* transducer *over* Σ *is an NFA over the alphabet* $\Sigma \times \Sigma$.

Definition 5.4 *Let* T *be a transducer over* Σ. *Given words* $w_1 = a_1 a_2 \ldots a_n$ *and* $w_2 = b_1 b_2 \ldots b_n$, *we say that* T accepts the pair (w_1, w_2) *if it accepts the word* $(a_1, b_1)...(a_n, b_n) \in (\Sigma \times \Sigma)^*$.

Definition 5.5 *Let* T *be a transducer.*

- T *accepts a pair* $(x, y) \in X \times X$ *if it accepts all encodings of* (x, y).
- T *rejects a pair* $(x, y) \in X \times X$ *if it accepts no encoding of* (x, y).
- T *recognizes a relation* $R \subseteq X \times X$ *if*

$$\mathcal{L}(T) = \{(w_x, w_y) \in (\Sigma \times \Sigma)^* \mid (w_x, w_y) \text{ encodes some pair of } R\} .$$

A relation is regular if it is recognized by some transducer.

- Examples of regular relations on numbers (*lsbf* encoding):
  - The identity relation $\{ (n, n) \mid n \in \mathbb{N}\}$
  - The relation $\{ (n, 2n) \mid n \in \mathbb{N}\}$

Example 5.6 The *Collatz function* is the function $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Determinism

- A transducer is deterministic if it is a DFA.

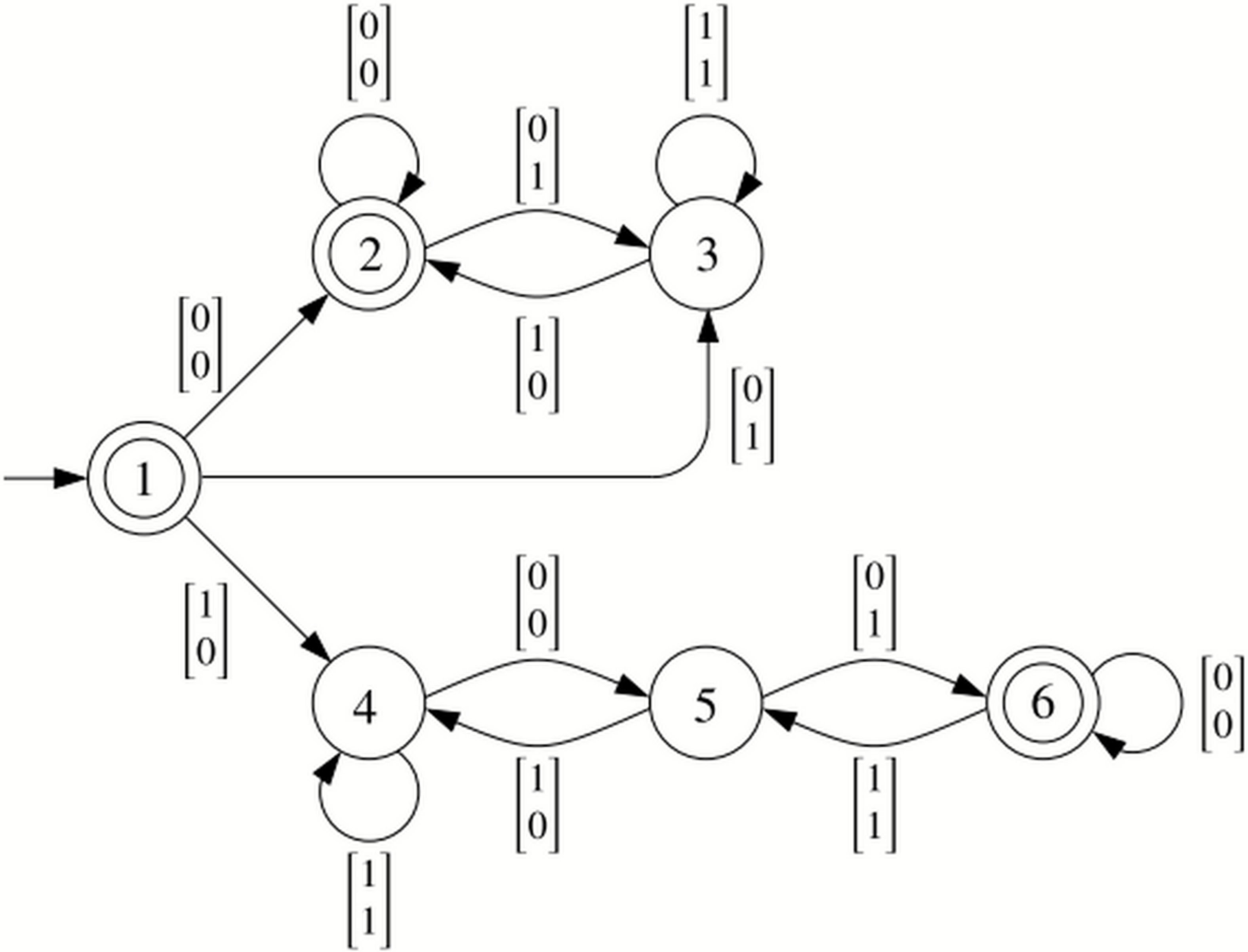
- Observe: if Σ has size n, then a state of a deterministic transducer with alphabet Σ x Σ has n^2 outgoing transitions.

- Warning! There is a different definition of determinism:
  - A letter $\begin{bmatrix} a \\ b \end{bmatrix}$ is interpreted as "output b on input a"
  - Deterministic transducer: only one move (and so only one output) for each input.

- Before implementing the new operations:
  - How do we check membership?
  - Can we compute union, intersection and complement of relations as for sets?

Implementing the operations

Projection



- Deleting the second component is not correct
  - Counterexample: $R = \{ (4,1) \}$
  - $s_{(4,1)} = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$
  - DFA for R:

Proj_1(T)

Input: transducer $T = (Q, \Sigma \times \Sigma, \delta, q_0, F)$

Output: NFA $A = (Q', \Sigma, \delta', q_0', F')$ with $\mathcal{L}(A) = \pi_1(\mathcal{L}(T))$

$Q' \leftarrow Q$; $q_0' \leftarrow q_0$; $F'' \leftarrow F$
$\delta' \leftarrow \emptyset$;
for all $(q, (a, b), q') \in \delta$ **do**
    add (q, a, q') **to** δ'
$F' \leftarrow PadClosure((Q', \Sigma, \delta', q_0', F''), \#)$

PadClosure$(A, \#)$

Input: NFA $A = (\Sigma \times \Sigma, Q, \delta, q_0, F)$

Output: new set F' of final states

$W \leftarrow F$; $F' \leftarrow \emptyset$;
while $W \neq \emptyset$ **do**
    pick q **from** W
    add q **to** F'
    for all $(q', \#, q) \in \delta$ **do**
        if $q' \notin F'$ **then add** q' **to** W
return F'

- Problem: we may be accepting $s_x\ \#^k\ \#^*$ instead of $s_x\ \#^*$ and so according to the definition we are not acepting x !

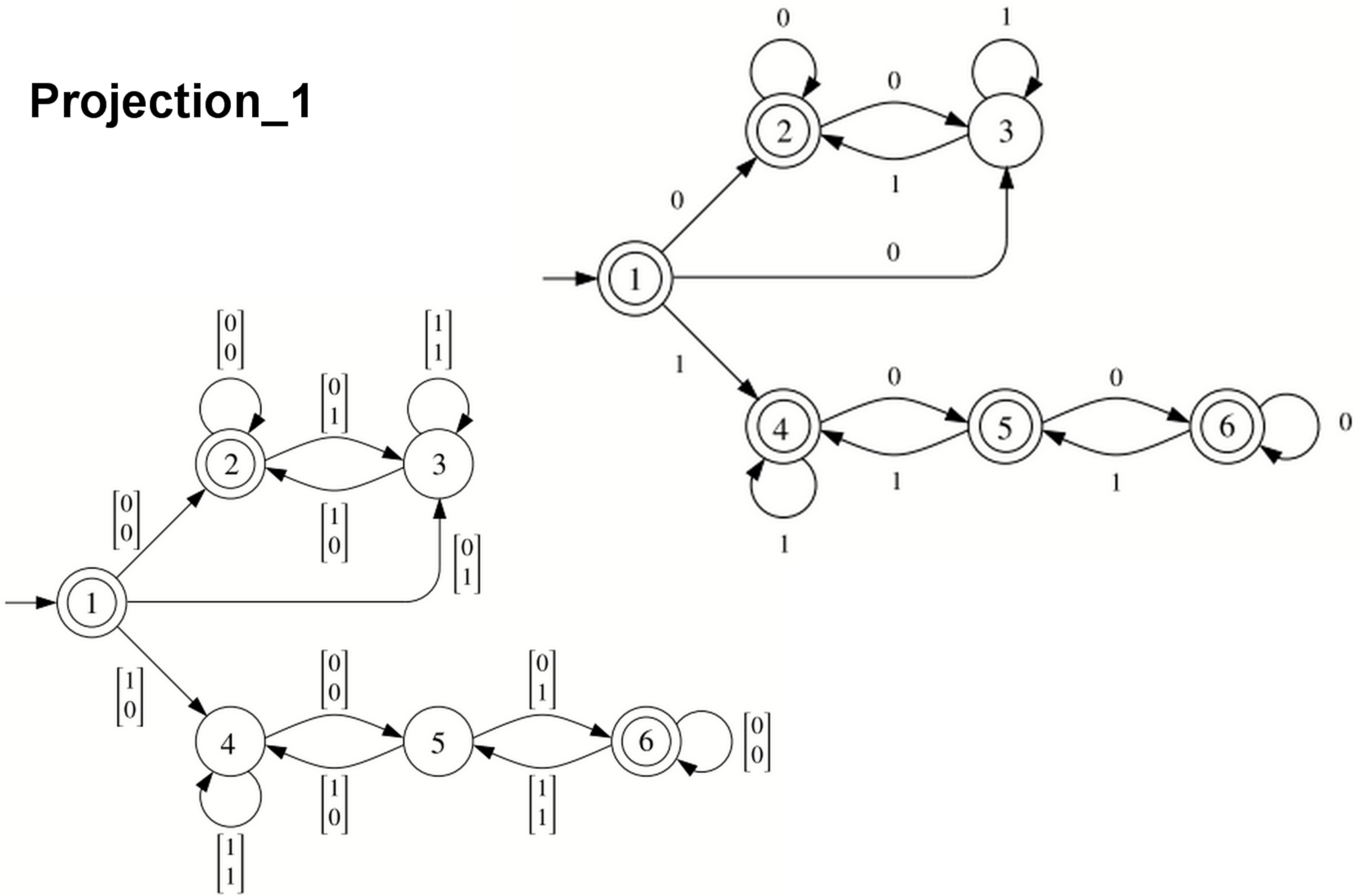
- Solution: if after eliminating the second components some non-final state goes with # ... # to a final state, we mark the state as final.

- Complexity: linear in the size of the transducer

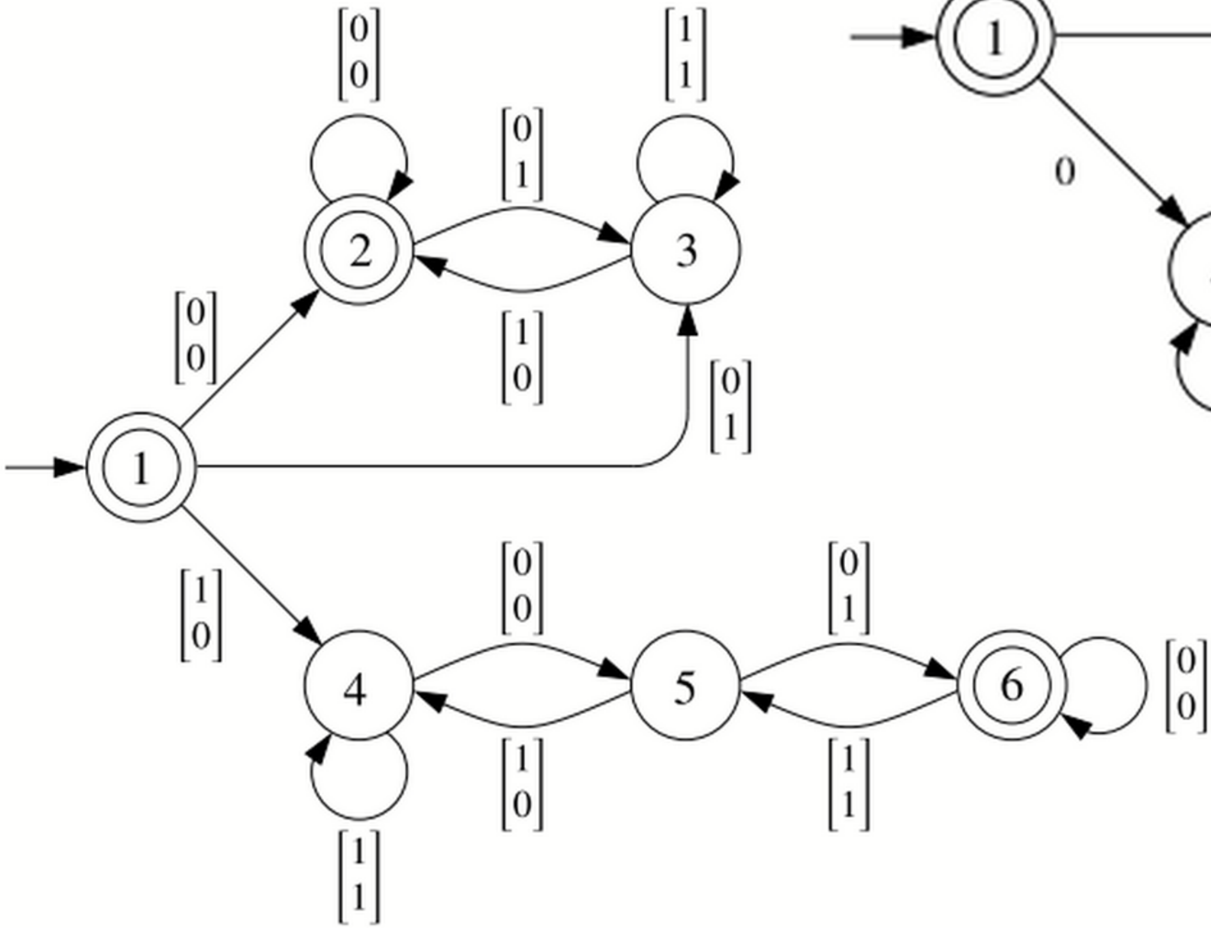
- Observe: the result of a projection may be a NFA, even if the transducer is deterministic!!

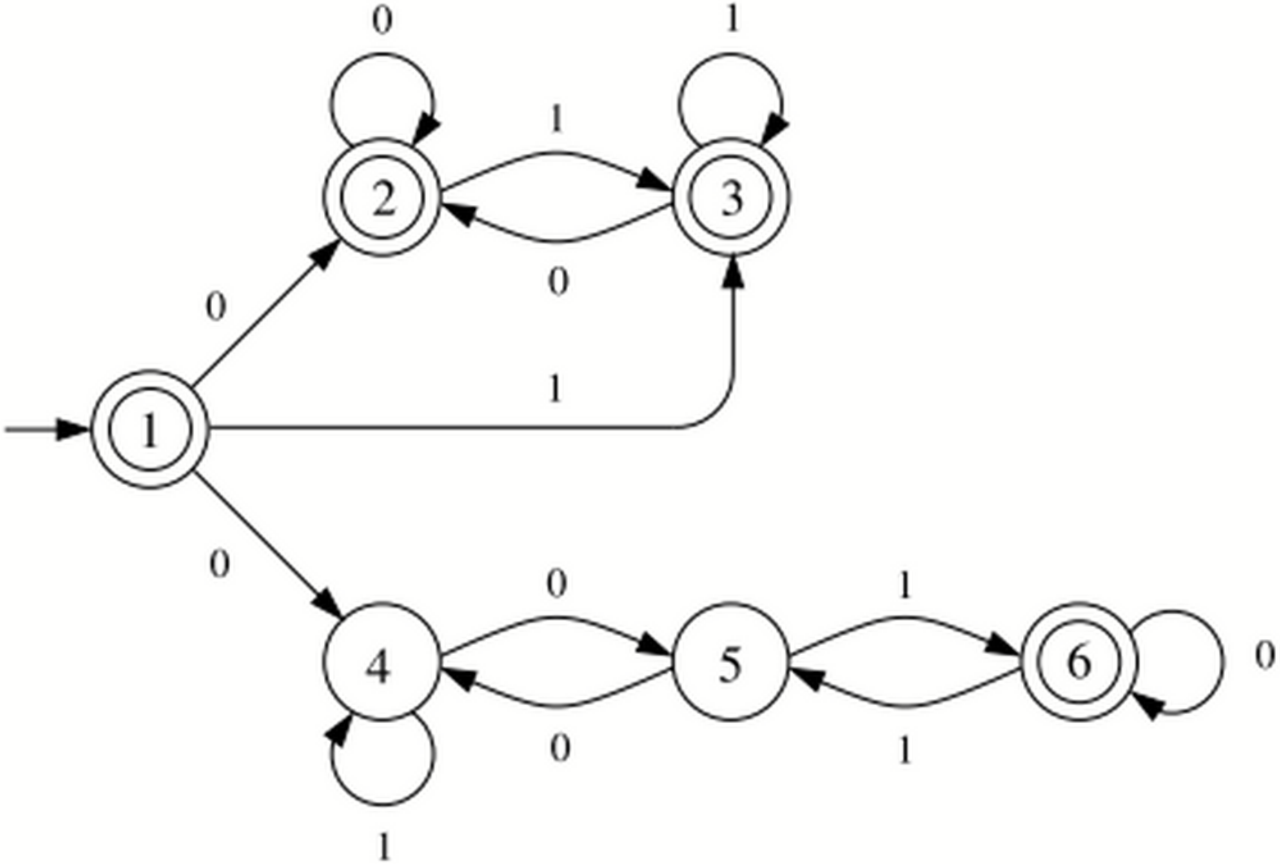
- This is the operation that prevents us from implementing all operations directly on DFAs.

Projection_1



Projection_2





Correctness proof

- Assume: transducer T recognizes a set of pairs
- Prove: the projection automaton A recognizes a set, and this set is the projection onto the first component of the set of pairs recognized by T.

a) A accepts either all encodings or no encoding of an object.

Assume A accepts at least one encoding w of an object x. We prove it accepts all.

If A accepts w, then T accepts $\begin{matrix} w \\ w' \end{matrix}$, for some w'. By assumption T accepts $\begin{matrix} w \\ w' \end{matrix} \begin{bmatrix} \# \\ \# \end{bmatrix}^*$, and so A accepts $w\,\#^*$. Moreover, $w = s_x \#^k$ for some $k > 0$, and so, by padding closure, A also accepts $s_x \#^j$ for every $j < k$.

b) A only accepts words that are encodings of objects.

Follows easily from the fact that T satisfies the same property for pairs of objects.

Correctness proof

c) If A accepts an object x , then there is an object y such that T accepts (x, y) .

$$
\begin{aligned}
& x \text{ accepted by } A \\
\Rightarrow \quad & s_x \text{ accepted by } A && \text{(part a)} \\
\Rightarrow \quad & \begin{matrix} s_x \\ w \end{matrix} \text{ accepted by } T \text{ for some } w
\end{aligned}
$$

By assumption, T only accepts pairs of words encoding some pair of objects. So w encodes some object y. By assumption, T then accepts all encodings of (x, y). So T accepts (x, y) .

Correctness proof

d) If a pair of objects (x, y) is accepted by T, then x is accepted by A.

(x, y) accepted by T

$\Rightarrow$ $\begin{matrix} w_x \\ w_y \end{matrix}$ accepted by T for some encodings w_x, w_y of x and y

$\Rightarrow$ w_x accepted by A

$\Rightarrow$ x accepted by A (part a))

Remember:

The projection automaton of a deterministic transducer may be nondeterministic.

Joi

- Goal: given transducers T_1, T_2 recognizing relations R_1, R_2, construct a transducer $T_1 \circ T_2$ recog∘nizing the relation $R_1 \circ R_2$.

- First step: construct a transducer T that accepts $\genfrac{}{}{0pt}{}{w}{v}$ iff there is a "connecting" word u such that

$\genfrac{}{}{0pt}{}{w}{u}$ is accepted by T_1 and $\genfrac{}{}{0pt}{}{u}{v}$ is accepted by T2.

We slightly modify the pairing construction.

Instead of:

$$\begin{bmatrix} q_{01} \\ q_{02} \end{bmatrix} \xrightarrow{a_1} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

iff

$$q_{01} \xrightarrow{a_1} q_{11}$$
$$q_{02} \xrightarrow{a_1} q_{12}$$

we now use

$$\begin{bmatrix} q_{01} \\ q_{02} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

iff

$$q_{01} \xrightarrow{\begin{bmatrix} a_1 \\ c_1 \end{bmatrix}} q_{11}$$
$$q_{02} \xrightarrow{\begin{bmatrix} c_1 \\ b_1 \end{bmatrix}} q_{12}$$

for some letter c1

The transducer T has a run

$$\begin{bmatrix} q_{01} \\ q_{02} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}} \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} q_{(n-1)1} \\ q_{(n-1)2} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_n \\ b_n \end{bmatrix}} \begin{bmatrix} q_{n1} \\ q_{n2} \end{bmatrix}$$

iff T_1 and T_2 have runs

$$\begin{array}{ccccccccc} q_{01} & \xrightarrow{\begin{bmatrix} a_1 \\ c_1 \end{bmatrix}} & q_{11} & \xrightarrow{\begin{bmatrix} a_2 \\ c_2 \end{bmatrix}} & q_{21} & \cdots & q_{(n-1)1} & \xrightarrow{\begin{bmatrix} a_n \\ c_n \end{bmatrix}} & q_{n1} \\ q_{02} & \xrightarrow{\begin{bmatrix} c_1 \\ b_1 \end{bmatrix}} & q_{12} & \xrightarrow{\begin{bmatrix} c_2 \\ b_2 \end{bmatrix}} & q_{22} & \cdots & q_{(n-1)2} & \xrightarrow{\begin{bmatrix} c_n \\ b_n \end{bmatrix}} & q_{n2} \end{array}$$

- We have the same problem as before.

- Let $R_1 = \{ (2,4) \}$, $\mathrm{R}_2 = \{ (4,2) \}$.

  Then $R_1 \circ R_2 = \{ (2,2) \}$.

- But the operation we have just defined does not yield the correct result.

- Solution: apply the padding closure again with padding symbol $\begin{bmatrix} \# \\ \# \end{bmatrix}$.

Join(T_1, T_2)

Input: transducers $T_1 = (Q_1, \Sigma \times \Sigma, \delta_1, q_{01}, F_1)$, $T_2 = (Q_2, \Sigma \times \Sigma, \delta_2, q_{02}, F_2)$

Output: transducer $T_1 \circ T_2 = (Q, \Sigma \times \Sigma, \delta, q_0, F)$

$Q, \delta, F' \leftarrow \emptyset;\ \ q_0 \leftarrow [q_{01}, q_{02}]$
$W \leftarrow \{[q_{01}, q_{02}]\}$
while $W \neq \emptyset$ **do**
    pick $[q_1, q_2]$ **from** W
    add $[q_1, q_2]$ **to** Q
    if $q_1 \in F_1$ and $q_2 \in F_2$ **then add** $[q_1, q_2]$ **to** F'
    for all $(q_1, (a, c), q'_1) \in \delta_1, (q_2, (c, b), q'_2) \in \delta_2$ **do**
        add $([q_1, q_2], (a, b), [q'_1, q'_2])$ **to** δ
        if $[q'_1, q'_2] \notin Q$ **then add** $[q'_1, q'_2]$ **to** W
$F \leftarrow$ **PadClosure**$((Q, \Sigma \times \Sigma\delta, q_0, F'), (\#, \#))$

Complexity: similar to pairing

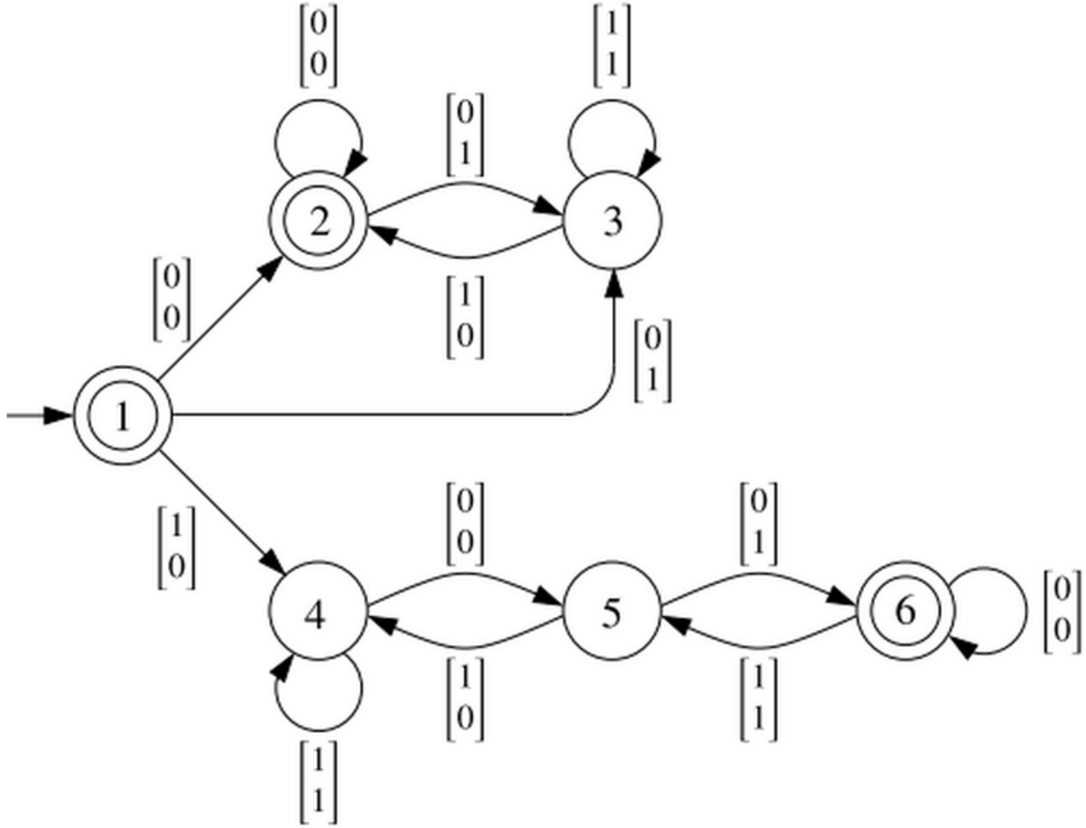
- Example:

  - Let f be the Collatz function.

  - Let $R_1 = R_2 = \{ (n, f(n)) \mid n \geq 0 \}$.

  - Then $R_1 \circ R_2 = \{ (n, f(f(n))) \mid n \geq 0 \}$.

$$f(f((n)) = \begin{cases} n/4 & \text{if } n \equiv 0 \, mod \, 4 \\ 3n/2 + 1 & \text{if } n \equiv 2 \, mod \, 4 \\ 3n/2 + 1/2 & \text{if } n \equiv 1 \, mod \, 4 \text{ or } n \equiv 3 \, mod \, 4 \end{cases}$$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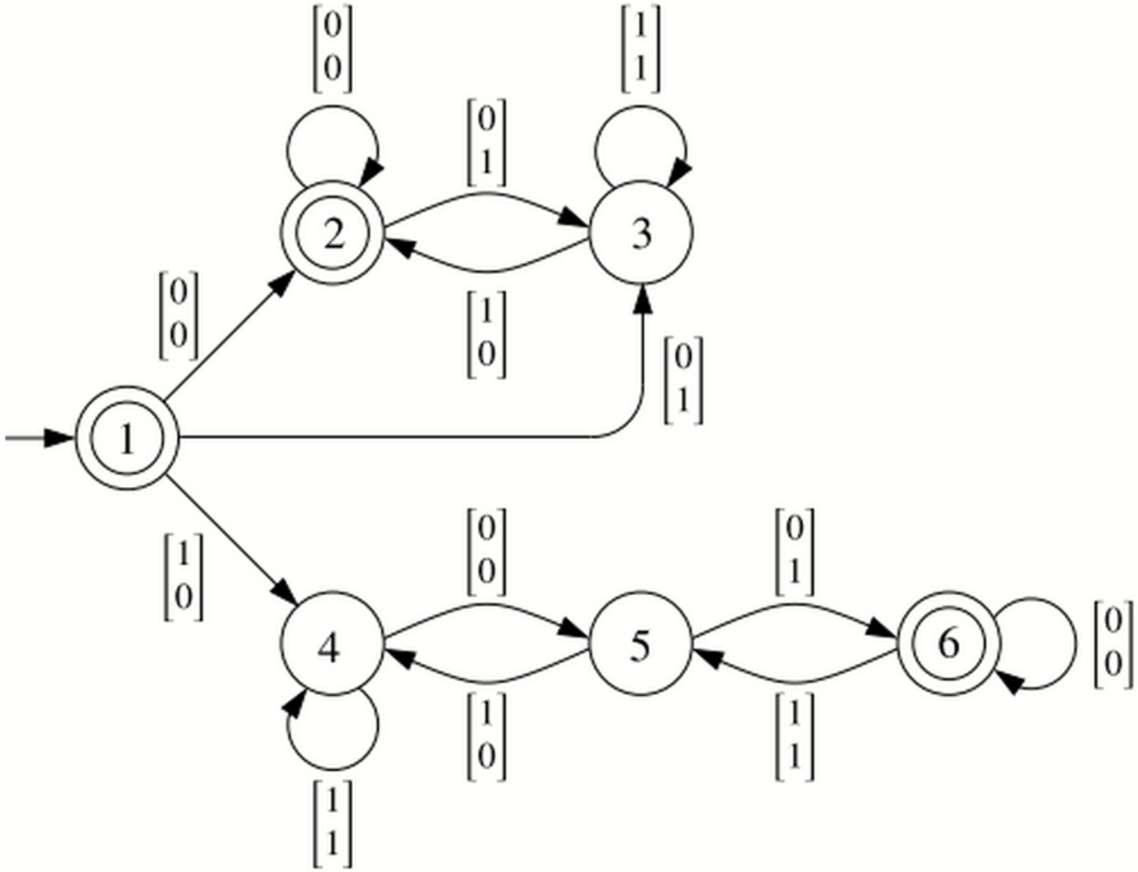
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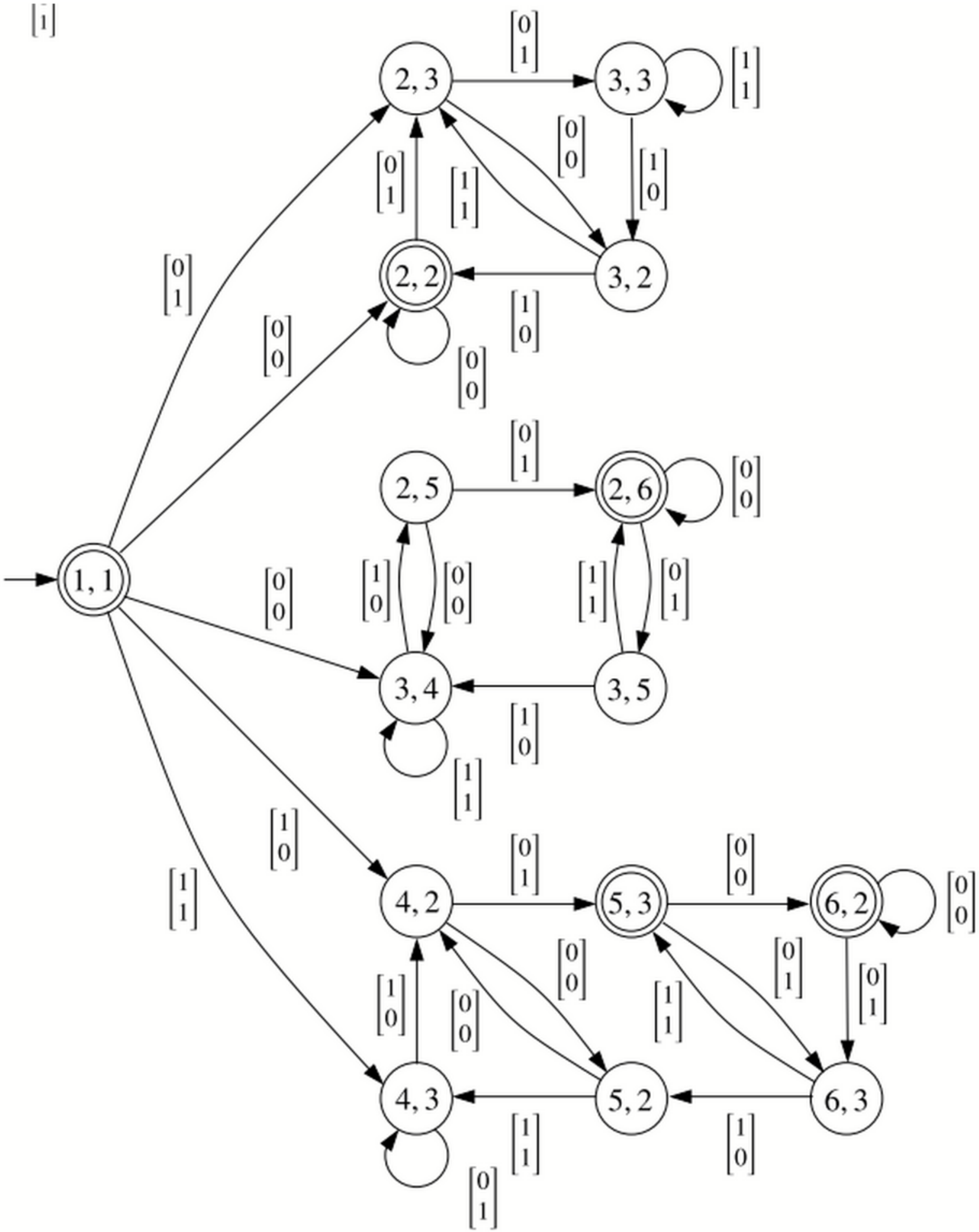

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Pre and Post

- Goal (for post):

  given

  – an automaton A recognizing a set X, and

  – a transducer T recognizing a relation R

  construct an automaton B recognizing the set

  $$\{ y \mid \exists\, x \in X : (x, y) \in R \}$$

  We slightly modify the construction for join.

Instead of:

$$\begin{bmatrix} q_{01} \\ q_{02} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \quad \text{iff} \quad q_{01} \xrightarrow{\begin{bmatrix} a_1 \\ c_1 \end{bmatrix}} q_{11}, \quad q_{02} \xrightarrow{\begin{bmatrix} c_1 \\ b_1 \end{bmatrix}} q_{12}$$

for some letter c1

we now use

$$\begin{bmatrix} q_{01} \\ q_{02} \end{bmatrix} \xrightarrow{b_1} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \quad \text{iff} \quad q_{01} \xrightarrow{a_1} q_{11}, \quad q_{02} \xrightarrow{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}} q_{12}$$

for some letter a_1

From Join to Post

Join(T_1, T_2)

Input: transducers $T_1 = (Q_1, \Sigma \times \Sigma, \delta_1, q_{01}, F_1)$, $T_2 = (Q_2, \Sigma \times \Sigma, \delta_2, q_{02}, F_2)$

Output: transducer $T_1 \circ T_2 = (Q, \Sigma \times \Sigma, \delta, q_0, F)$

$Q, \delta, F' \leftarrow \emptyset$; $q_0 \leftarrow [q_{01}, q_{02}]$

$W \leftarrow \{[q_{01}, q_{02}]\}$

while $W \neq \emptyset$ **do**

  pick $[q_1, q_2]$ **from** W

  add $[q_1, q_2]$ **to** Q

  if $q_1 \in F_1$ and $q_2 \in F_2$ **then add** $[q_1, q_2]$ **to** F'

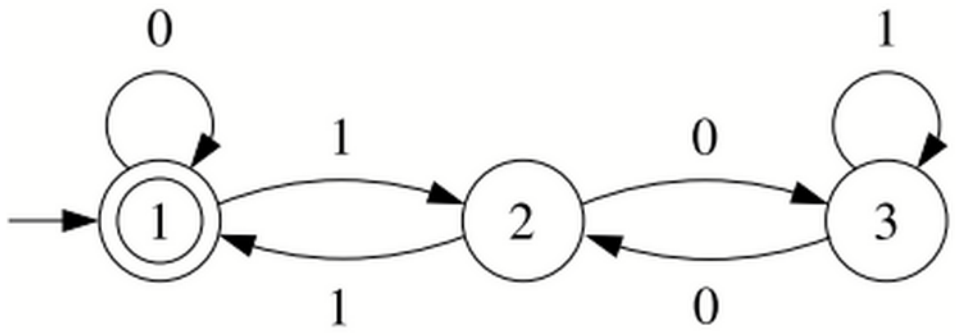
  for all $(q_1, (a, c), q_1') \in \delta_1, (q_2, (c, b), q_2') \in \delta_2$ **do**

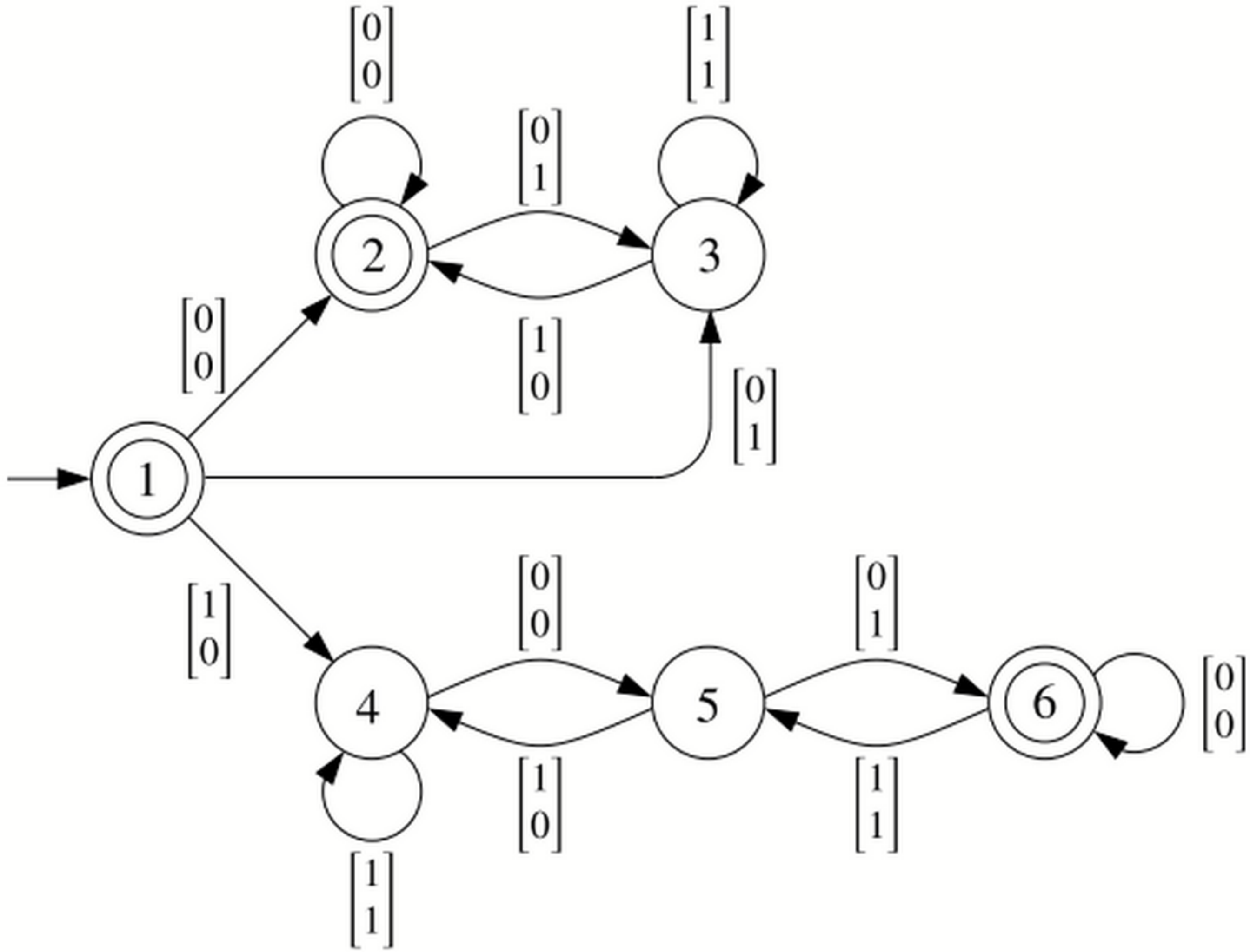
    add $([q_1, q_2], (a, b), [q_1', q_2'])$ **to** δ

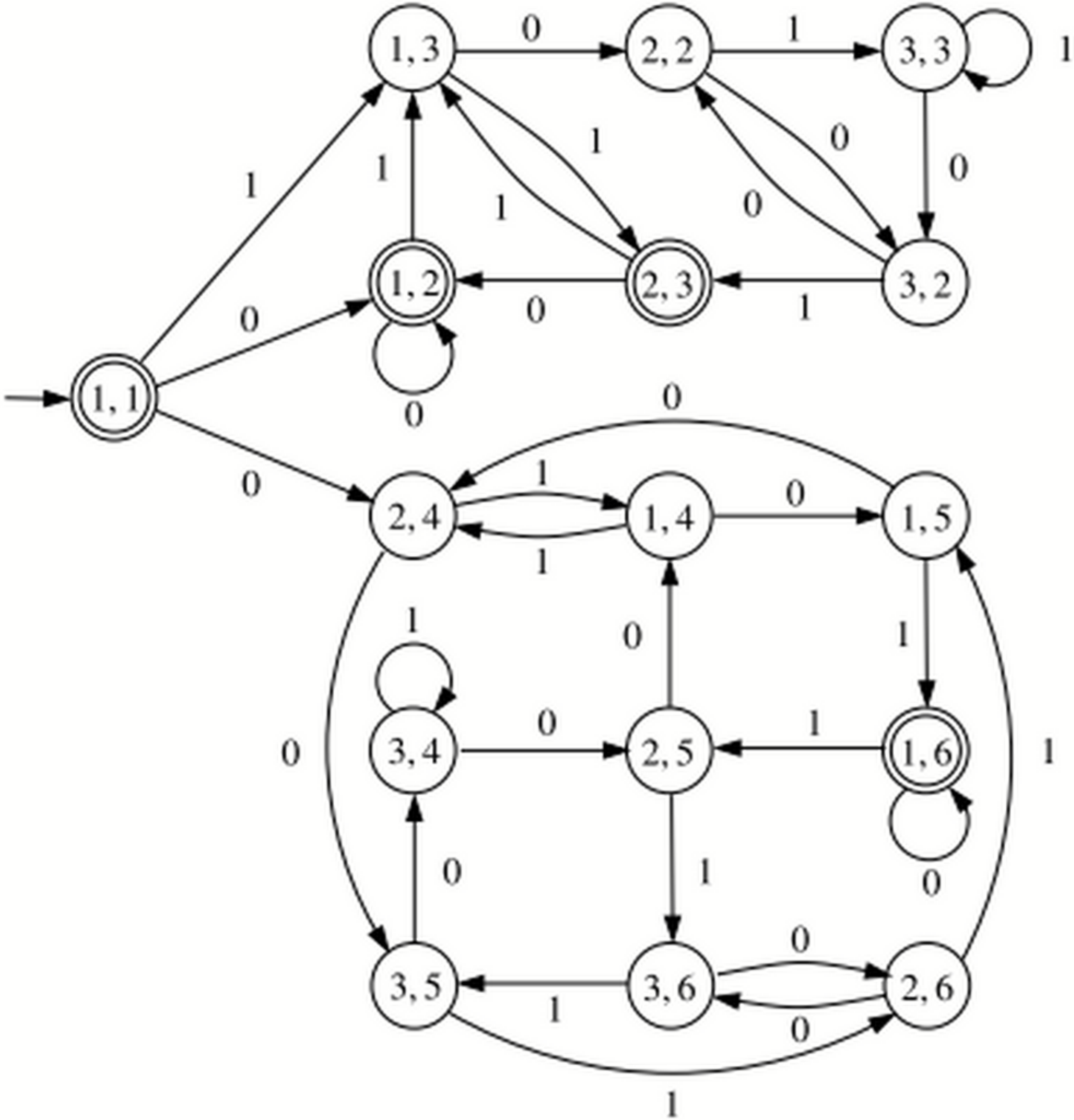
    if $[q_1', q_2'] \notin Q$ **then add** $[q_1', q_2']$ **to** W

$F \leftarrow$ **PadClosure**$((Q, \Sigma \times \Sigma \delta, q_0, F'), (\#, \#))$

Example: compute the set { f(n) | n multiple of 3 }



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