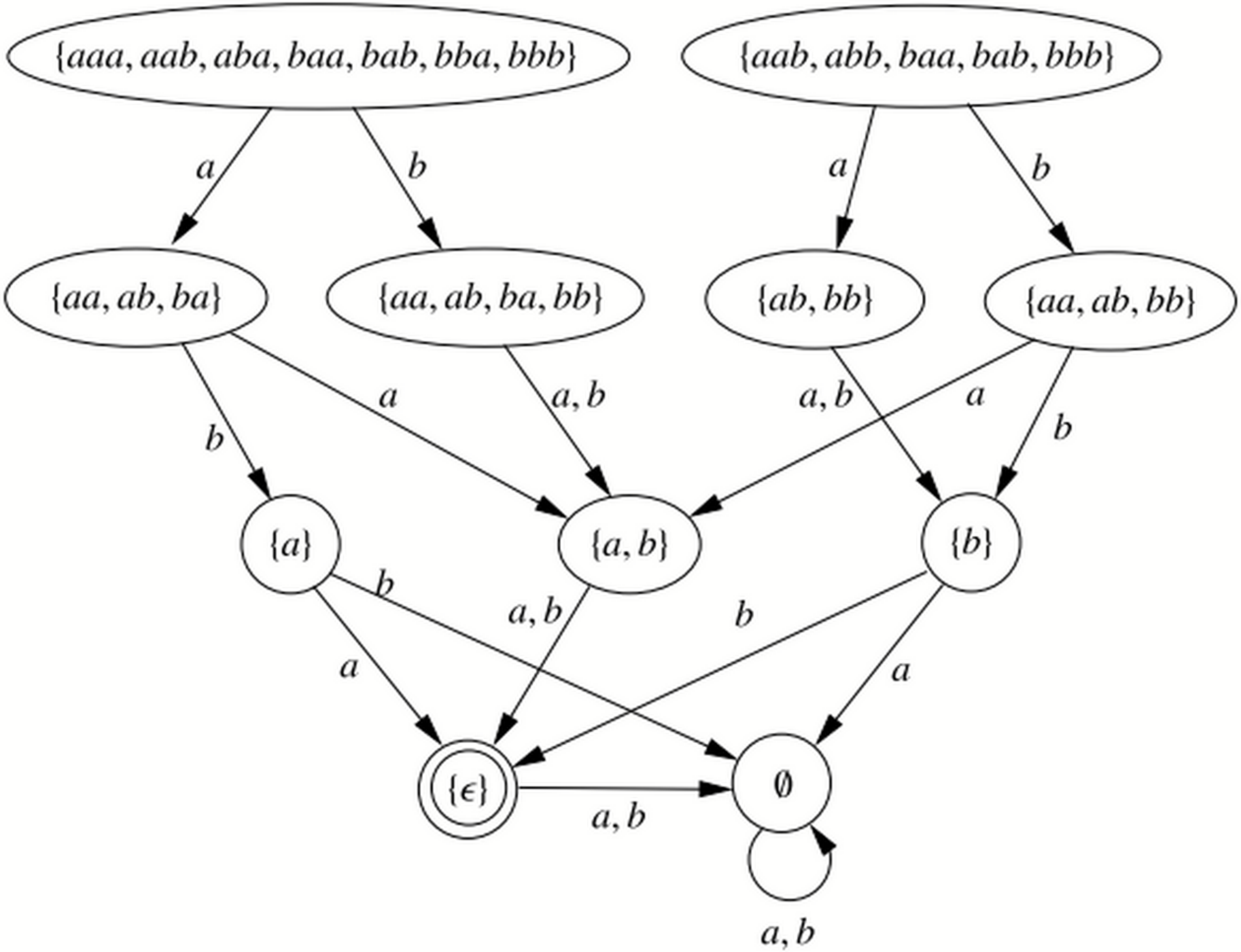
# Finite Universes

- When the universe is finite (e.g., the interval $[0, 2^{32} - 1]$ ), all objects can be encoded by words of the same length.
- A language $L$ has length $n \geq 0$ if
  - $L = \emptyset$ and $n = 0$, or
  - $L \neq \emptyset$ and every word of $L$ has length $n$.
- $L$ is a fixed-length language if it has length $n$ for some $n \geq 0$ .
- Observe:
  - Fixed-length languages contain finitely many words.
  - $\emptyset$ and $\{\varepsilon\}$ are the only two languages of length 0.

# The Master Automaton



- The master automaton over $\Sigma$ is the tuple $\mathrm{M} = (Q_M, \Sigma, \delta_M, F_M)$, where
  - $Q_M$ is the set of all fixed-length languages;
  - $\delta_M\colon\ Q_M \times\ \Sigma \to Q_M$ is given by $\delta_M(L, a) = L^a$;
  - $F_M$ is the set $\{\ \{\varepsilon\}\ \}$.

- Prop: The language recognized from state $L$ of the master automaton is $L$.

  Proof: By induction on the length $n$ of $L$.

  $n = 0$. Then either $L = \emptyset$ or $L = \{\varepsilon\}$ , and result follows by inspection.

  $n > 0$. Then $\delta_M(L, a) = L^a$ for every $a \in \Sigma$, and $L^a$ has smaller length than $L$. By induction hypothesis the state $L^a$ recognizes the language $L^a$, and so the state $L$ recognizes the language $L$.

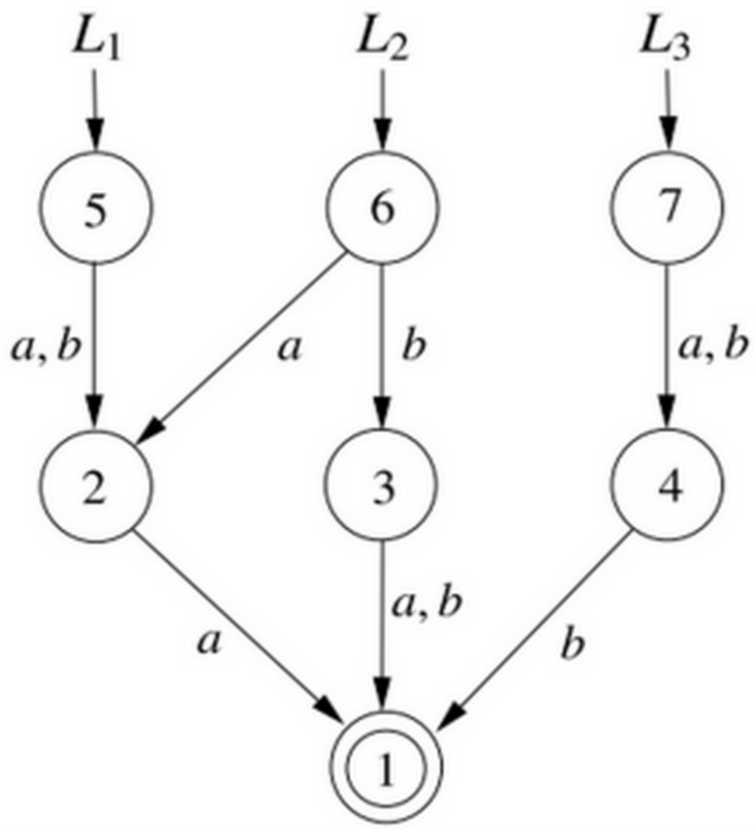
- We denote the „fragment“ of the master automaton reachable from state $L$ by $A_L$ :
  - Initial state is $L$.
  - States and transitions are those reachable from $L$.

- Prop: $A_L$ is the minimal DFA recognizing $L$.

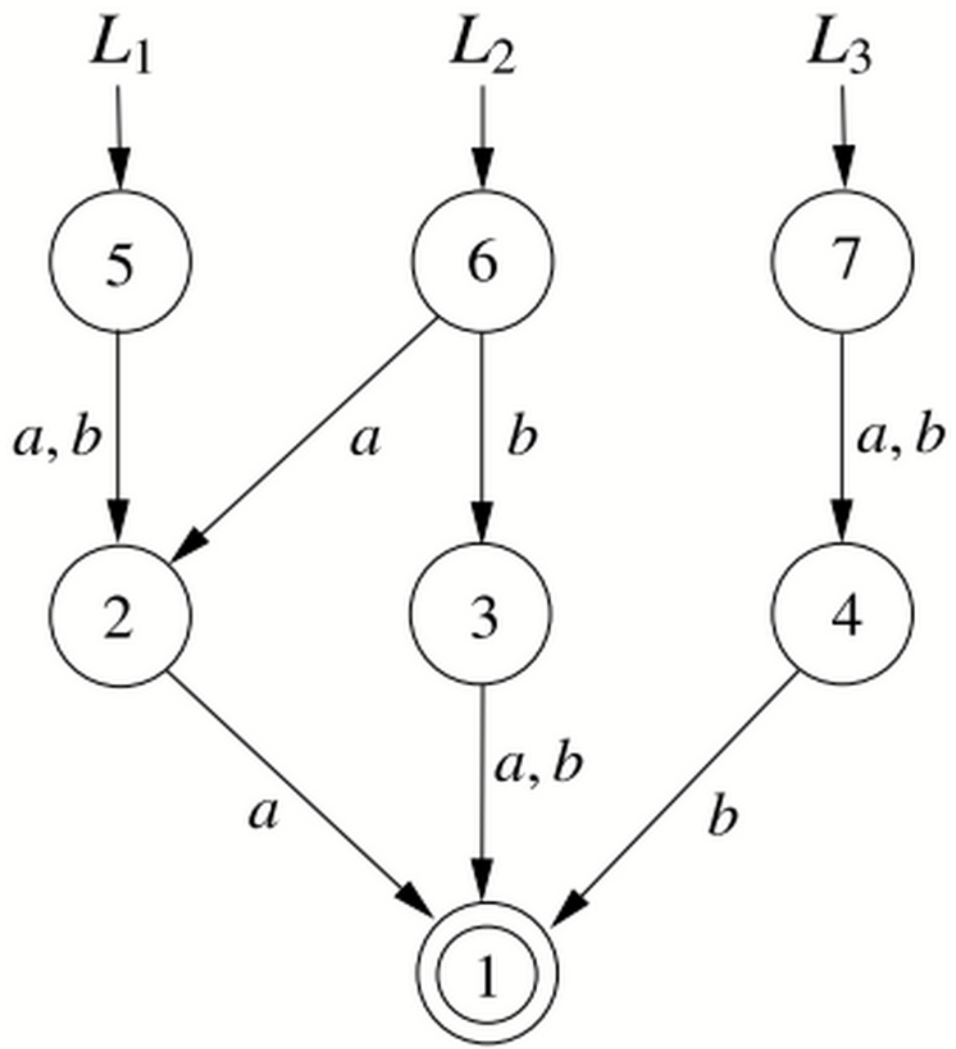
  Proof: By definition, all states of $A_L$ are reachable from its initial state. Since every state of the master automaton recognizes its „own“ language, distinct states of $A_L$ recognize distinct languages.

# Data structure for fixed-length languages

- The structure representing the set of languages $\mathcal{L} = \{L_1, \ldots, L_m\}$ is the fragment of the master automaton containing states $L_1, \ldots, L_m$ and their descendants.
- It is a multi-DFA , i.e., a DFA with multiple initial states.

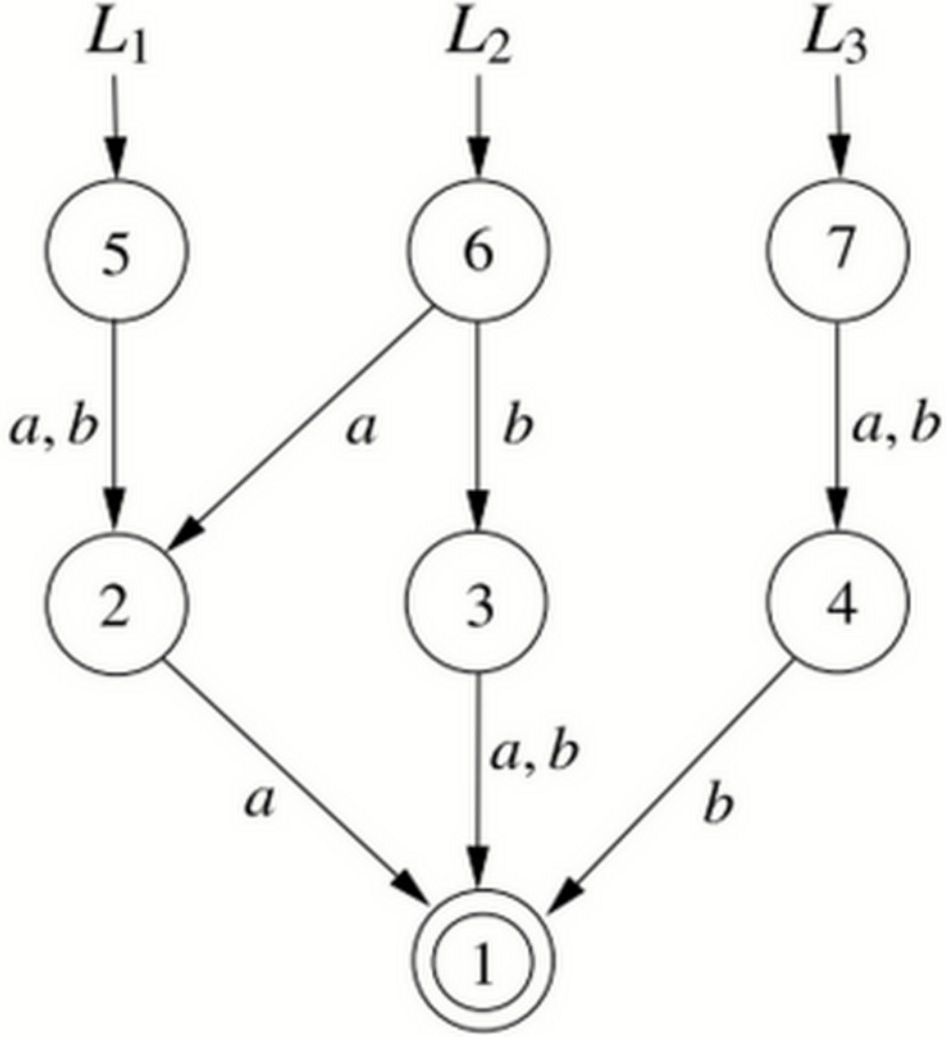


In order to manipulate multi-DFAs we represent them as a *table of nodes*. Assume $\Sigma = \{a_1, \ldots, a_m\}$. A *node* is a pair $\langle q, s \rangle$, where $q$ is a *state identifier* and $s = (q_1, \ldots, q_m)$ is the *successor tuple* of the node. The multi-DFA is represented by a table containing a node for each state, but the state corresponding to the empty language[1].




| Ident. | $a$-succ | $b$-succ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 2 | 0 |
| 6 | 2 | 4 |
| 7 | 3 | 0 |

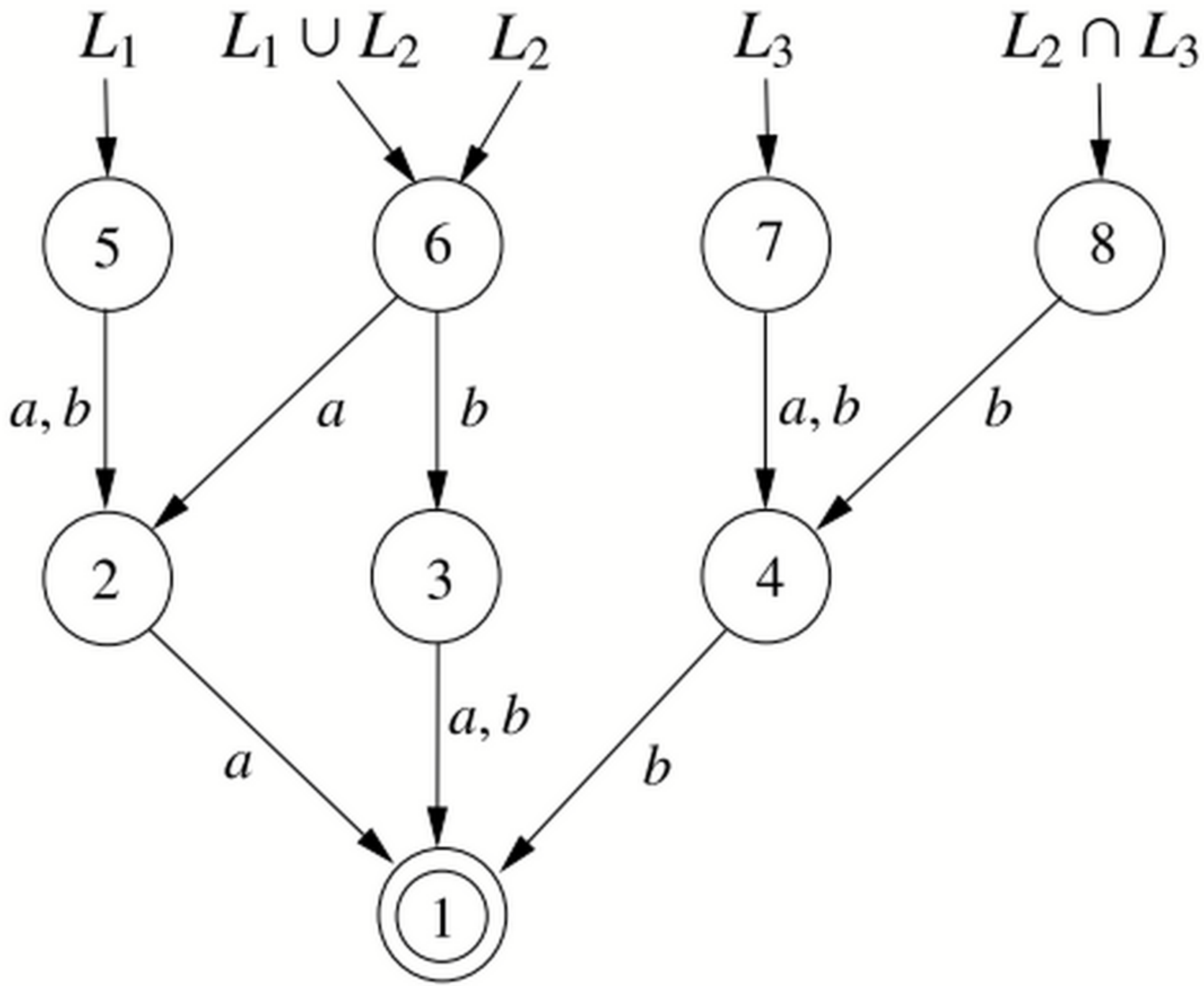
- We represent multi-DFAs as tables of nodes .
- A node is a pair $\langle q, s \rangle$ where
  - $q$ is a state identifier, and
  - $s = (q_1, \ldots, q_m)$ is a successor tuple.
- The table for a multi-DFA contains a node for each state but the state for the empty language.




| Ident. | $a$-succ | $b$-succ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 2 | 0 |
| 6 | 2 | 4 |
| 7 | 3 | 0 |

- The procedure $make[T](s)$
  - returns the state identifier of the node of table $T$ having s as successor tuple, if such a node exists;
  - otherwise it adds a new node $\langle q, s\rangle$ to $T$, where $q$ is a fresh identifier , and returns $q$.
- $make[T](s)$ assumes that $T$ contains a node for every identifier in $s$.

# Implementing union and intersection



- We give a recursive algorithm $inter[T](q_1, q_2)$:
  - Input: state identifiers $q_1, q_2$ from table $T$.
  - Output: identifier of the state recognizing $L(q_1) \cap L(q_2)$ in the multi-DFA for $T$.
  - Side-effect: if the identifier is not in $T$, then the algorithm adds new nodes to $T$, i.e., after termination the table T may have been extended.
- The algorithm follows immediately from the following properties

  (1) if $L_1 = \emptyset$, then $L_1 \cap L_2 = \emptyset$ ;

  (2) if $L_2 = \emptyset$, then $L_1 \cap L_2 = \emptyset$ ;

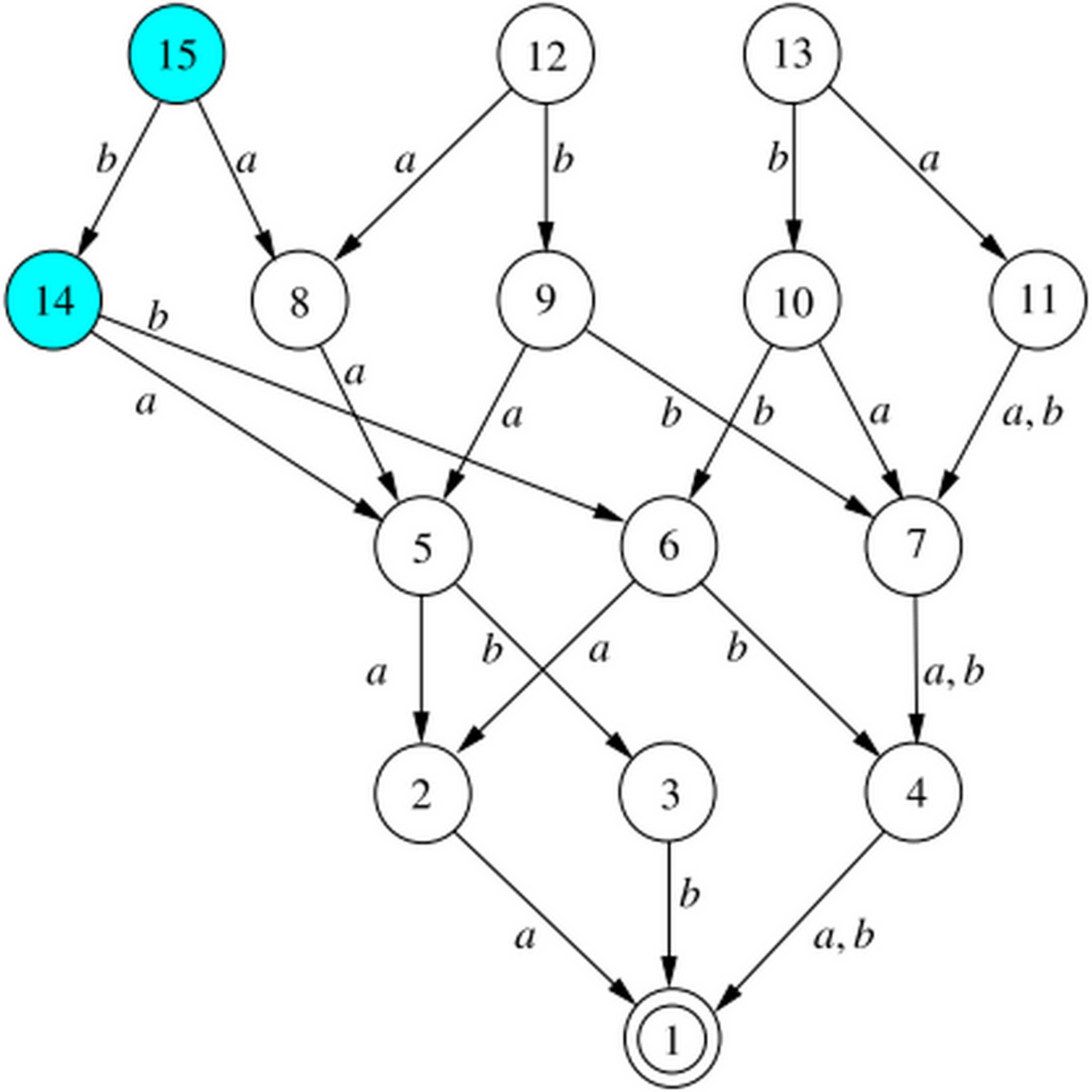
  (3) If $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$ , then $(L_1 \cap L_2)^a = L_1^a \cap L_2^a$ for every $a \in \Sigma$.

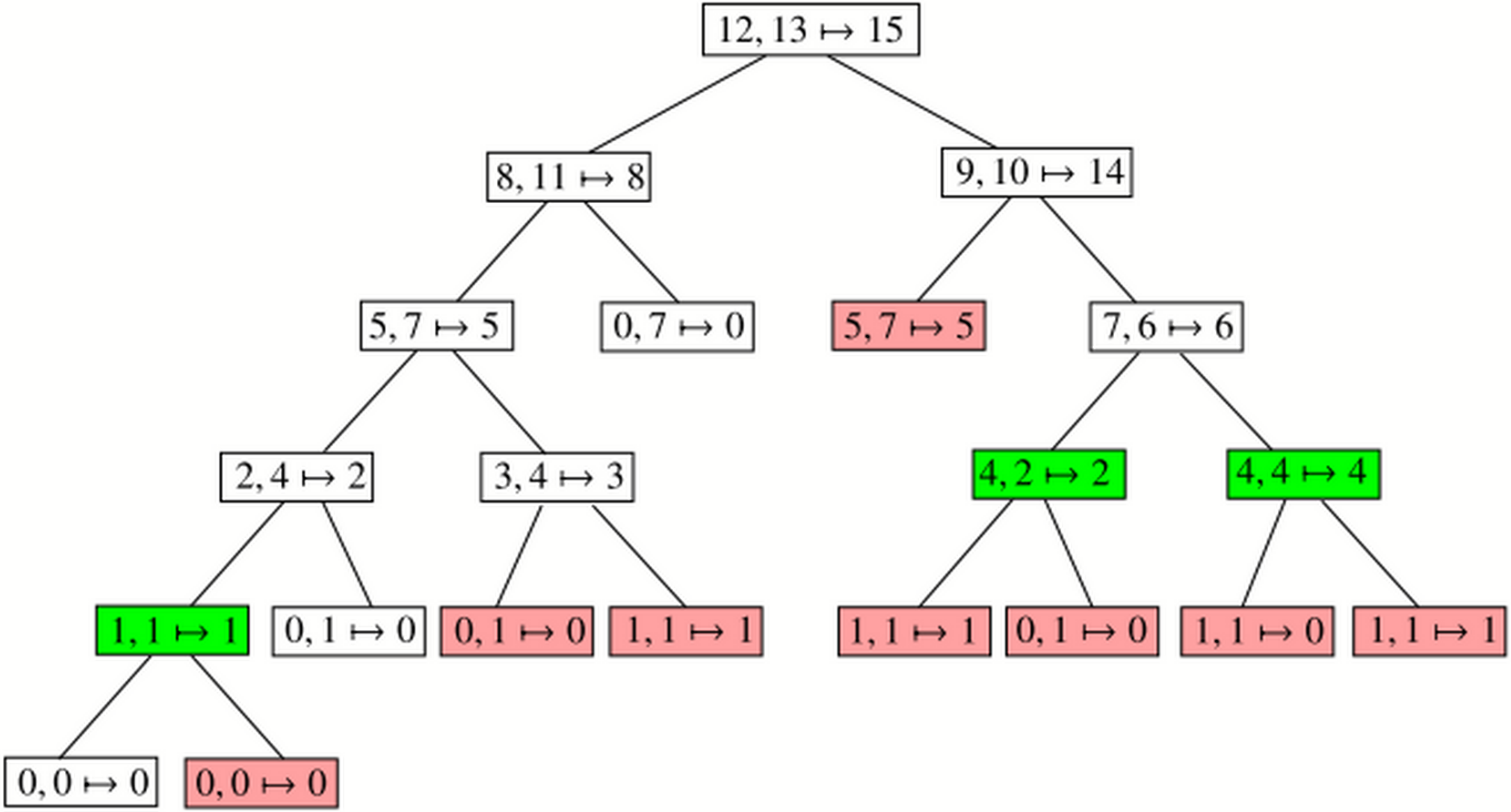
$inter[T](q_1, q_2)$

**Input:** table $T$, states $q_1, q_2$ of $T$

**Output:** state recognizing $\mathcal{L}(q_1) \cap \mathcal{L}(q_2)$

**if** $G(q_1, q_2)$ is not empty **then return** $G(q_1, q_2)$<br>
**if** $q_1 = q_\emptyset \vee q_2 = q_\emptyset$ **then return** $q_\emptyset$<br>
**if** $q_1 \neq q_\emptyset \wedge q_2 \neq q_\emptyset$ **then**<br>
    **for all** $i = 1, \ldots, m$ **do** $r_i \leftarrow inter[T](q_1^{a_i}, q_2^{a_i})$<br>
    $G(q_1, q_2) \leftarrow \texttt{make}[T](r_1, \ldots, r_m)$<br>
    **return** $G(q_1, q_2)$


15
12
13
b
a
a
b
b
a
14
8
9
10
11
b
a
a
a
b
b
a
a, b
5
6
7
a
b
a
b
a, b
2
3
4
a
b
a, b
1



12, 13 ↦ 15
8, 11 ↦ 8
9, 10 ↦ 14
5, 7 ↦ 5
0, 7 ↦ 0
5, 7 ↦ 5
7, 6 ↦ 6
2, 4 ↦ 2
3, 4 ↦ 3
4, 2 ↦ 2
4, 4 ↦ 4
1, 1 ↦ 1
0, 1 ↦ 0
0, 1 ↦ 0
1, 1 ↦ 1
1, 1 ↦ 1
0, 1 ↦ 0
1, 1 ↦ 0
1, 1 ↦ 1
0, 0 ↦ 0
0, 0 ↦ 0

# Fixed-length complement

In principle ill-defined, because the complement of a fixed-length language is not fixed-length.

We implement the fixed-length complement instead.

Can't we just swap the states for the empty language and the language containing the empty word?

Yes and no ...

# Fixed-length complement

**Equations:**

- if $L = \emptyset$, then $\overline{L} = \Sigma^n$, where $n$ is the length of $L$;
- if $L = \{\epsilon\}$, then $\overline{L} = \emptyset$; and
- if $\emptyset \neq L \neq \{\epsilon\}$, then $\left(\overline{L}\right)^a = \overline{L^a}$.

  (Observe that $w \in \left(\overline{L}\right)^a$ iff $aw \notin L$ iff $w \notin L^a$ iff $w \in \overline{L^a}$.)

$comp[T, n](q)$

**Input:** table $T$, length $n$, state $q$ of $T$ of length $n$

**Output:** state recognizing the fixed-length complement of $L(q)$

**if** $G(q)$ is not empty **then return** $G(q)$
**if** $n = 0$ **and** $q = q_\emptyset$ **then return** $q_\epsilon$
**else if** $n = 0$ **and** $q = q_\epsilon$ **then return** $q_\emptyset$
**else** $/ * \ n \geq 1 \ * /$
  **for all** $i = 1, \ldots, m$ **do** $r_i \leftarrow comp[T, n-1](q^{a_i})$
  $G(q) \leftarrow \texttt{make}[T](r_1, \ldots, r_m)$
  **return** $G(q)$

# Emptiness

$empty[T](q)$
**Input:** table $T$, state $q$ of $T$
**Output:** **true** if $\mathcal{L}(q) = \emptyset$, **false** otherwise

1 **return** $q = q_\emptyset$

# Universality

- if $L = \emptyset$, then $L$ is not universal;
- if $L = \{\epsilon\}$, then $L$ is universal;
- if $\emptyset \neq L \neq \{\epsilon\}$, then $L$ is universal iff $L^a$ is universal for every $a \in \Sigma$.

$univ[T](q)$

**Input:** table $T$, state $q$ of $T$

**Output:** **true** if $\mathcal{L}(q)$ is fixed-length universal,
**false** otherwise

**if** $G(q)$ is not empty **then return** $G(q)$
**if** $q = q_\emptyset$ **then return false**
**else if** $q = q_\epsilon$ **then return true**
**else** $/ * \; q \neq q_\emptyset$ and $q \neq q_\epsilon \; * /$
    **for all** $i = 1, \ldots, m$ **do** $r_i \leftarrow comp[T](q^{a_i})$
    $G(q) \leftarrow$ **and**$(univ[T](r_1), \ldots, univ[T](r_m))$
    **return** $G(q)$

# Inclusion and Equality

**Inclusion.** Given two languages $L_1, L_2 \subseteq \Sigma^n$, in order to check $L_1 \subseteq L_2$ we compute $L_1 \cap L_2$ and check whether it is equal to $L_1$ using the equality check shown next. The complexity is dominated by the complexity of computing the intersection.

$eq[T](q_1, q_2)$
**Input:** table $T$, states $q_1, q_2$ of $T$
**Output:** **true** if $\mathcal{L}(q_1) = \mathcal{L}(q_2)$, **false** otherwise
1 **return** $q_1 = q_2$

$eq[T_1, T_2](q_1, q_2)$

**Input:** tables $T_1$, $T_2$, states $q_1$ of $T_1$, $q_2$ of $T_2$

**Output:** **true** if $\mathcal{L}(q_1) = \mathcal{L}(q_2)$, **false** otherwise

**if** $G(q_1, q_2)$ is not empty **then return** $G(q_1, q_2)$<br>
**if** $q_1 = q_{\emptyset 1}$ and $q_2 = q_{\emptyset 2}$ **then** $G(q_1, q_2) \leftarrow$ `true`<br>
**else if** $q_1 = q_{\emptyset 1}$ and $q_2 \neq q_{\emptyset 2}$ **then** $G(q_1, q_2) \leftarrow$ `false`<br>
**else if** $q_1 \neq q_{\emptyset 1}$ and $q_2 = q_{\emptyset 2}$ **then** $G(q_1, q_2) \leftarrow$ `false`<br>
**else** $/ * q_1 \neq q_{\emptyset 1}$ and $q_2 \neq q_{\emptyset 2} * /$<br>
    $G(q_1, q_2) \leftarrow$ **and**(`eq`$(q_1^{a_1}, q_2^{a_1}), \ldots,$ `eq`$(q_1^{a_m}, q_2^{a_m})$)<br>
**return** $G(q_1, q_2)$

# What if the starting point is an NFA?

- Given: NFA $A$ accepting a fixed-length language and containing no cycles.

  Goal: simultaneously determinize and minimize $A$
- Each state of $A$ accepts a fixed-length language.
- We give an algorithm *state*($S$):
  - Input: a subset $S$ of states of $A$ accepting languages of the same length.
  - Output: the state of the master automaton accepting $\bigcup_{q \in S} L(q)$.
- Goal is achieved by calling *state*($\{q_0\}$)

Equations:

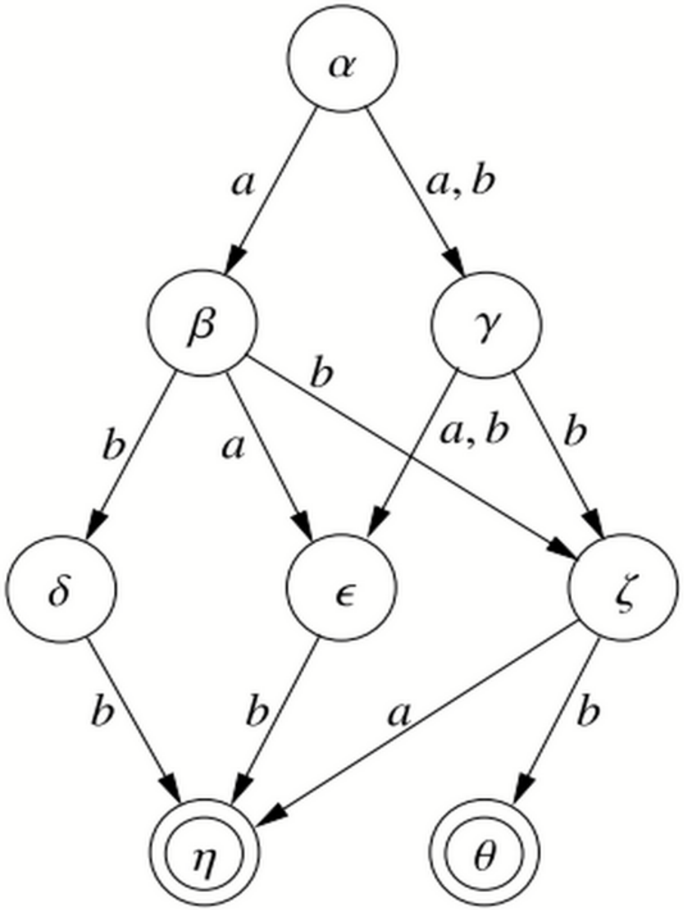
- if $S = \emptyset$ then $\mathcal{L}(S) = \emptyset$;
- if $S \cap F \neq \emptyset$ then $\mathcal{L}(S) = \{\epsilon\}$
- if $S \neq \emptyset$ and $S \cap F = \emptyset$, then $\mathcal{L}(S) = \bigcup_{i=1}^{n} a_i \cdot \mathcal{L}(S_i)$, where $S_i = \delta(S, a_i)$.

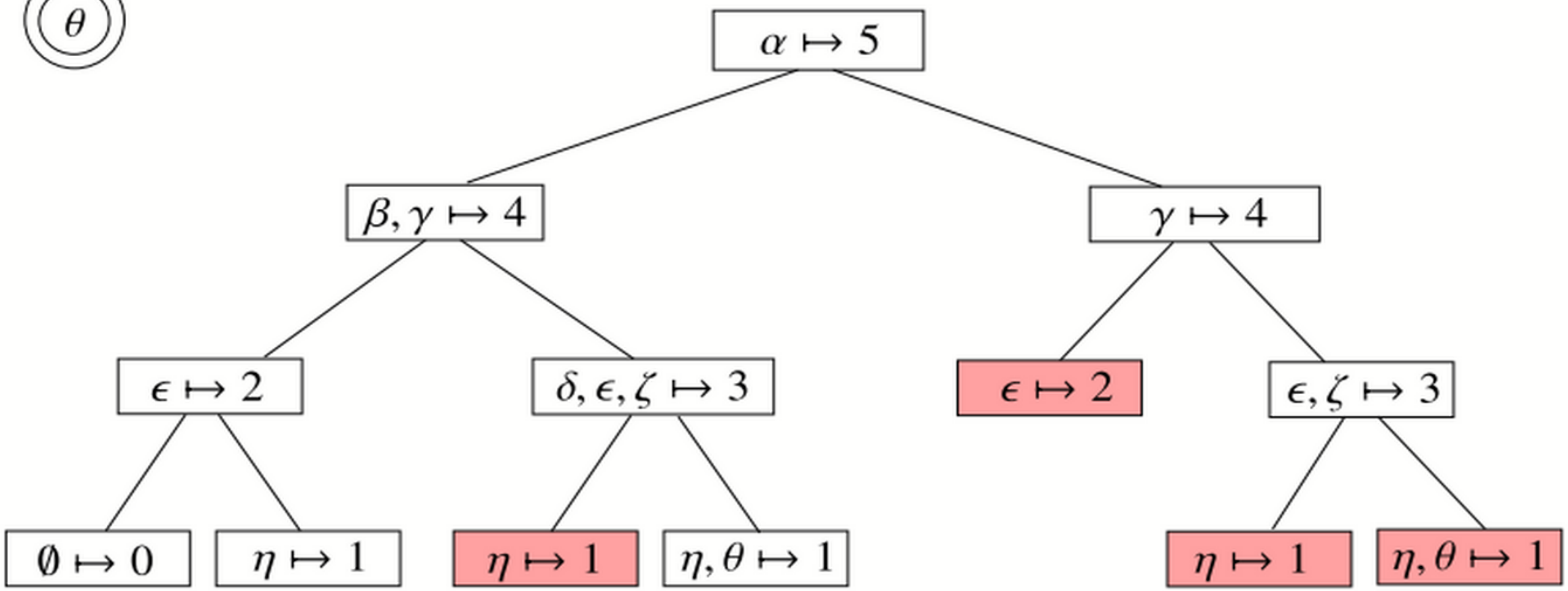
*state*[$A$]($S$)

**Input:** NFA $A = (Q, \Sigma, \delta, q_0, F)$, set $S \subseteq Q$

**Output:** master state recognizing $\mathcal{L}(S)$

```
if G(S) is not empty then return G(S)
else if S = ∅ then return q_∅
else if S ∩ F ≠ ∅ then return q_ϵ
else  /* S ≠ ∅ and S ∩ F = ∅ */
    for all i = 1, ..., m do S_i ← δ(S, a_i)
  G(S) ← make(state[A](S_1), ..., state[A](S_m));
  return G(S)
```


α
a
a, b
β
γ
b
b
a
a, b
b
δ
ϵ
ζ
b
b
a
b
η
θ


α ↦ 5
β, γ ↦ 4
γ ↦ 4
ϵ ↦ 2
δ, ϵ, ζ ↦ 3
ϵ ↦ 2
ϵ, ζ ↦ 3
∅ ↦ 0
η ↦ 1
η ↦ 1
η, θ ↦ 1
η ↦ 1
η, θ ↦ 1

# Operations on relations

**Definition 6.10** *A word relation $R \subseteq \Sigma^* \times \Sigma^*$ has* length $n \geq 0$ *if it is empty and $n = 0$, or if it is nonempty and for all pairs $(w_1, w_2)$ of $R$ the words $w_1$ and $w_2$ have length $n$. If $R$ has length $n$ for some $n \geq 0$, then we say that $R$ is a* fixed-length *word relation, or that $R$* has fixed-length.

**Definition 6.12** *The* master transducer *over the alphabet $\Sigma$ is the tuple $MT = (Q_M, \Sigma \times \Sigma, \delta_M, F_M)$, where*

- *$Q_M$ is is the set of all fixed-length relations;*
- *$\delta_M : Q_M \times (\Sigma \times \Sigma) \to Q_M$ is given by $\delta_M(R, [a, b]) = R^{[a,b]}$ for every $q \in Q_M$ and $a, b \in \Sigma$;*
- *$F_M = \{(\varepsilon, \varepsilon)\}$.*

With T_R as the "fragment" of MT with R as root we get:

**Proposition 6.13** *For every fixed-length word relation $R$, the transducer $T_R$ is the minimal deterministic transducer recognizing $R$.*

# Storing minimal transducers

Like minimal DFA, minimal deterministic transducers are represented as tables of nodes. However, a remark is in order: since a state of a deterministic transducer has $|\Sigma|^2$ successors, one for each letter of $\Sigma \times \Sigma$, a row of the table has $|\Sigma|^2$ entries, too large when the table is only sparsely filled. Sparse transducers over $\Sigma \times \Sigma$ are better encoded as NFAs over $\Sigma$ by introducing auxiliary states: a transition $q \xrightarrow{[a,b]} q'$ of the transducer is "simulated" by two transitions $q \xrightarrow{a} r \xrightarrow{b} q'$, where $r$ is an auxiliary state with exactly one input and one output transition.

# Computing joins

Equations:

- $\emptyset \circ R = R \circ \emptyset = \emptyset$;
- $\{(\varepsilon, \varepsilon)\} \circ \{(\varepsilon, \varepsilon)\} = \{(\varepsilon, \varepsilon)\}$;
- $R_1 \circ R_2 = \bigcup_{a,b,c \in \Sigma} [a, b] \cdot \left(R_1^{[a,c]} \circ R_2^{[c,b]}\right)$.

**Input:** transducer table $T$, states $q_1, q_2$ of $T$

**Output:** state recognizing $\mathcal{L}(q_1) \circ \mathcal{L}(q_2)$

$join[T](q_1, q_2)$<br>
    **if** $G(q_1, q_2)$ is not empty **then return** $G(q_1, q_2)$<br>
    **if** $q_1 = q_\emptyset$ **or** $q_2 = q_\emptyset$ **then return** $q_\emptyset$<br>
    **else if** $q_1 = q_\epsilon$ **and** $q_2 = q_\epsilon$ **then return** $q_\epsilon$<br>
    **else** $/ * \; q_\emptyset \neq q_1 \neq q_\epsilon, q_\emptyset \neq q_2 \neq q_\epsilon \; * /$<br>
        **for all** $(a_i, a_j) \in \Sigma \times \Sigma$ **do**<br>
            $q_{a_i,a_j} \leftarrow union[T]\left(join\left(q_1^{[a_i,a_1]}, q_2^{[a_1,a_j]}\right), \ldots, join\left(q_1^{[a_i,a_m]}, q_2^{[a_m,a_j]}\right)\right)$<br>
        $G(q_1, q_2) = make(q_{a_1,a_1}, \ldots, q_{a_1,a_m}, \ldots, q_{a_m,a_m})$<br>
        **return** $G(q_1, q_2)$

# Pre and Post

Pre and Post can be reduced to intersection and projection. Define:

$$emb(L) = \{[v_1, v_2] \in (\Sigma \times \Sigma)^n \mid v_2 \in L\}$$

$$pre_S(L) = \{w_1 \in \Sigma^n \mid \exists [v_1, v_2] \in S : v_1 = w_1 \text{ and } v_2 \in L\}$$

Then we have:

$$pre_S(L) \quad = \quad proj_1(S \cap emb(L))$$

We use this to derive equations.

# Equations:

$$\text{if } S = \emptyset \text{ or } L = \emptyset, \text{ then } pre_S(L) = \emptyset;$$

$$\text{if } S \neq \emptyset \neq L \text{ then } pre_S(L) = \bigcup_{a,b \in \Sigma} a \cdot pre_{S[a,b]}(L^b),$$

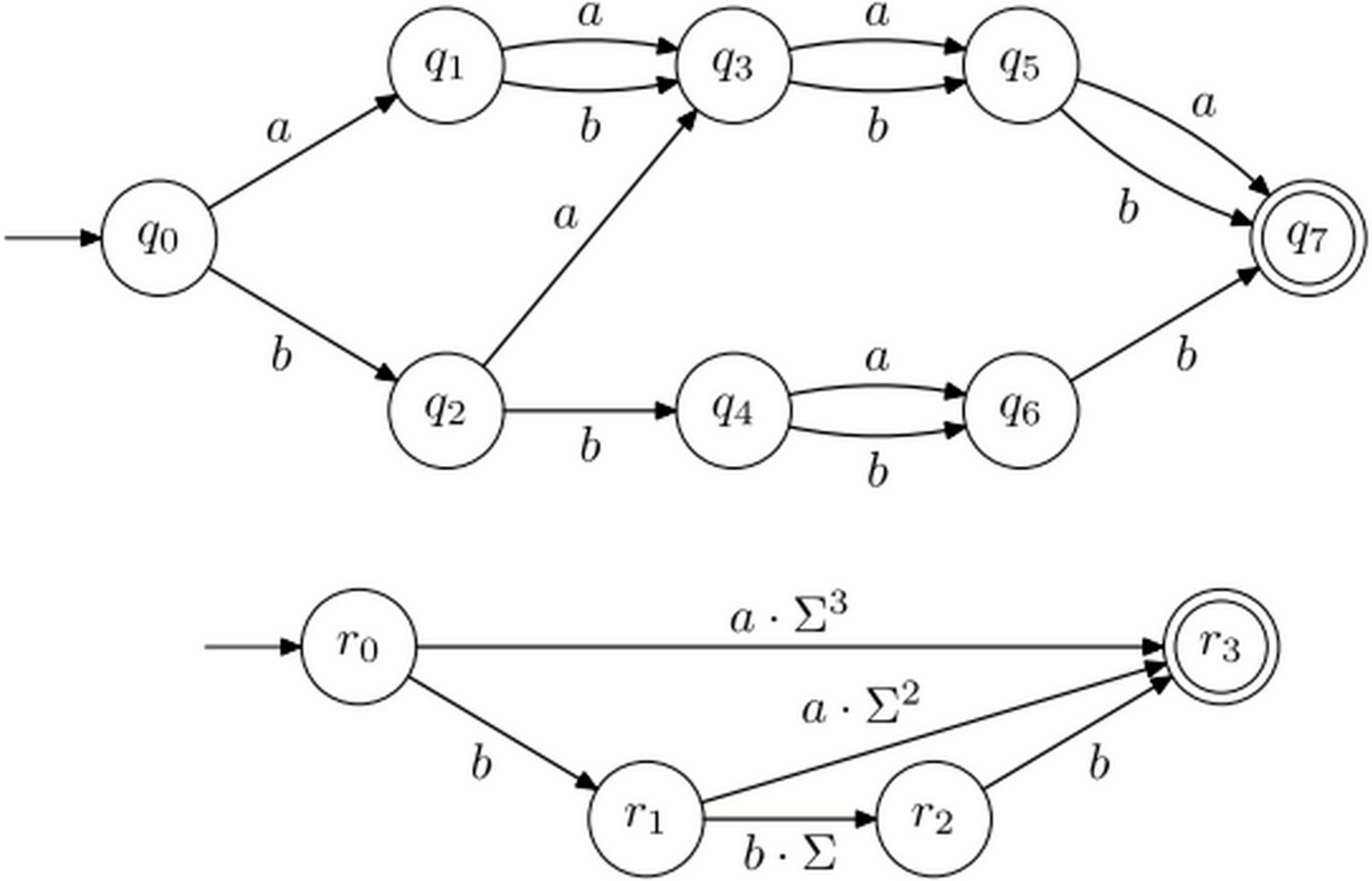
$$\text{where } S^{[a,b]} = \{w \in (\Sigma \times \Sigma)^* \mid [a,b]w \in S\}.$$

$$
\begin{aligned}
(pre_S(L))^a &= (proj_1(S \cap emb(L)))^a \\
&= \left( proj_1 \left( \bigcup_{b \in \Sigma} [a,b] \cdot (S \cap emb(L))^{[a,b]} \right) \right)^a \\
&= \left( \bigcup_{b \in \Sigma} proj_1 \left( [a,b] \cdot (S \cap emb(L))^{[a,b]} \right) \right)^a \\
&= \left( \bigcup_{b \in \Sigma} a \cdot proj_1 \left( (S \cap emb(L))^{[a,b]} \right) \right)^a \\
&= \bigcup_{b \in \Sigma} proj_1 \left( (S \cap emb(L))^{[a,b]} \right) \\
&= \bigcup_{b \in \Sigma} proj_1 \left( S^{[a,b]} \cap emb \left( L^b \right) \right) \\
&= \bigcup_{b \in \Sigma} pre_{S^{[a,b]}}(L^b)
\end{aligned}
$$

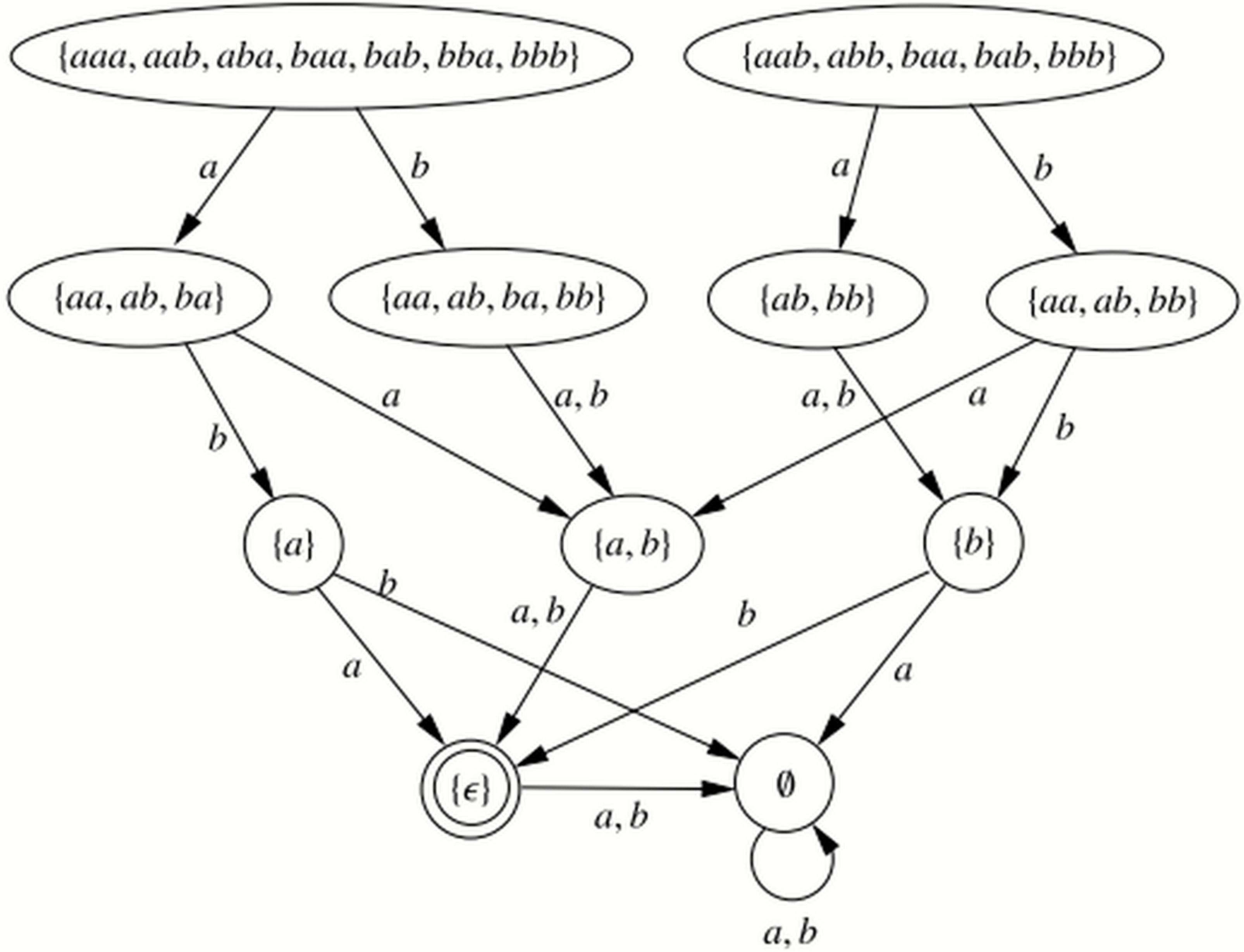
**Input:** transducer table $TT$, table $T$, state $r$ of $TT$, state $q$ of $T$
**Output:** state of $T$ recognizing $pre_{\mathcal{L}(r)}(\mathcal{L}(q))$

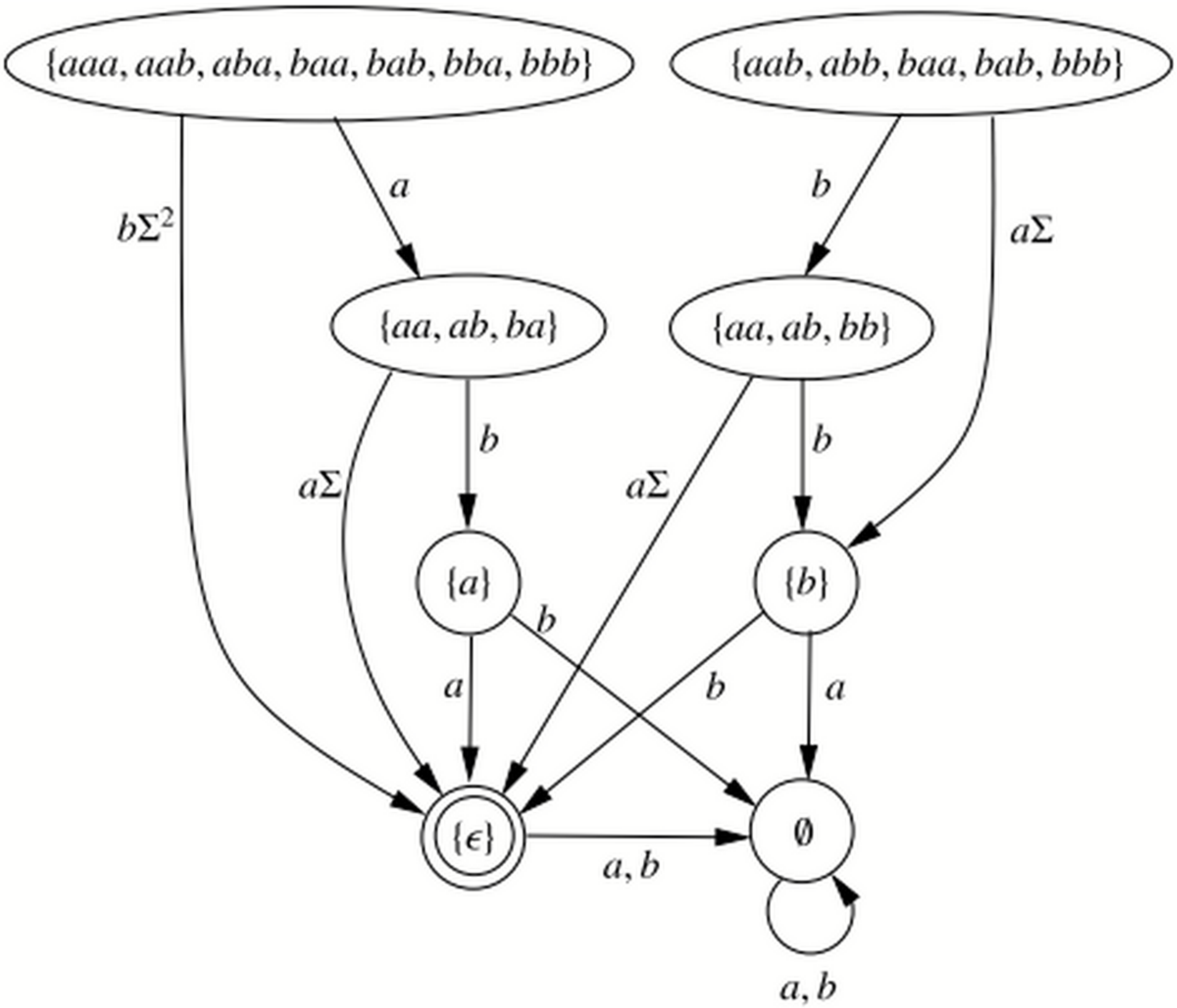
$pre[TT, T](r, q)$
  **if** $G(r, q)$ is not empty **then return** $G(r, q)$
  **if** $r = r_\emptyset$ **or** $q = q_\emptyset$ **then return** $q_\emptyset$
  **else if** $r = r_\epsilon$ **and** $q = q_\epsilon$ **then return** $q_\epsilon$
  **else**
    **for all** $a_i \in \Sigma$ **do**
      $q_{a_i} \leftarrow union\left(pre[TT, T]\left(q^{[a_i, a_1]}, r^{a_1}\right), \ldots, pre[TT, T]\left(q^{[a_i, a_m]}, r^{a_m}\right)\right)$
    $G(r, q) \leftarrow make(q_{a_1}, \ldots, q_{a_m});$
    **return** $G(r, q)$

# Binary Decision



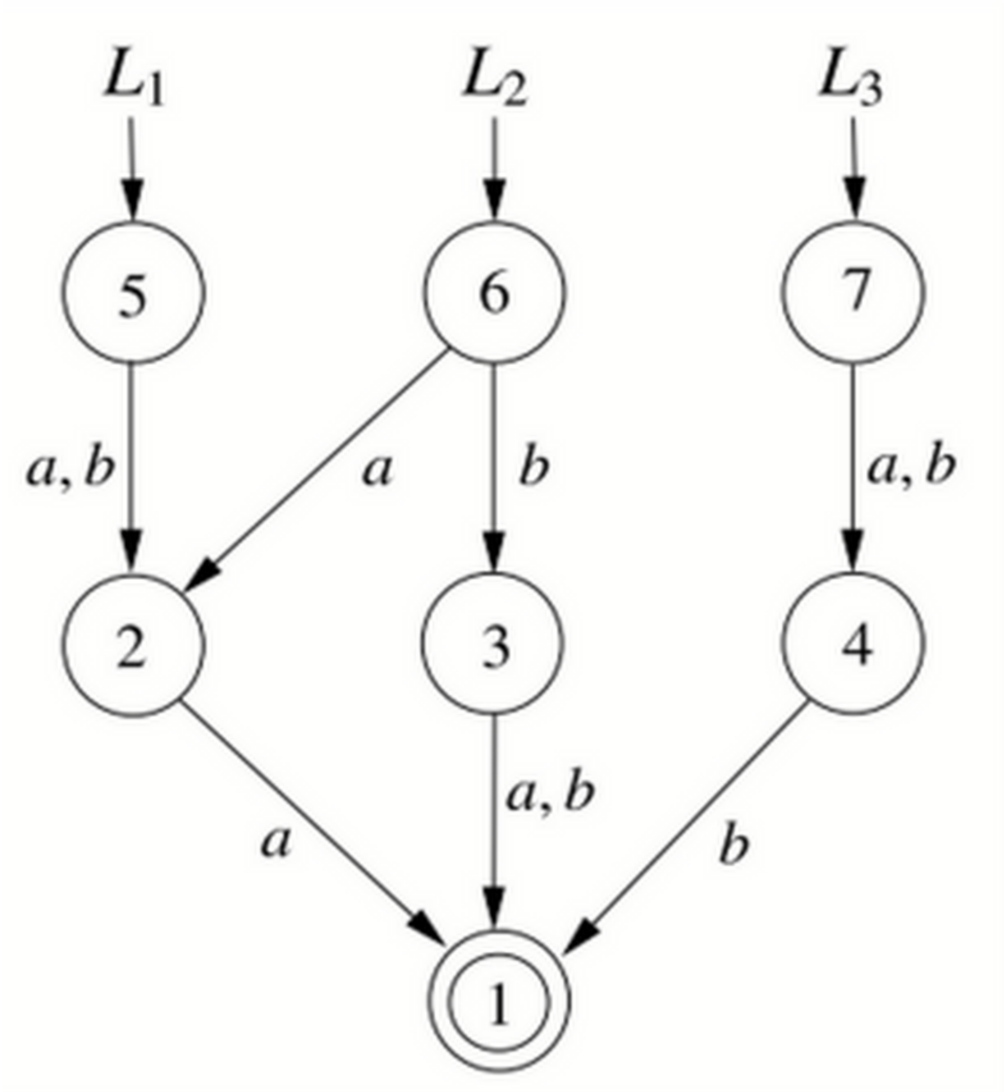
# The master z-automaton

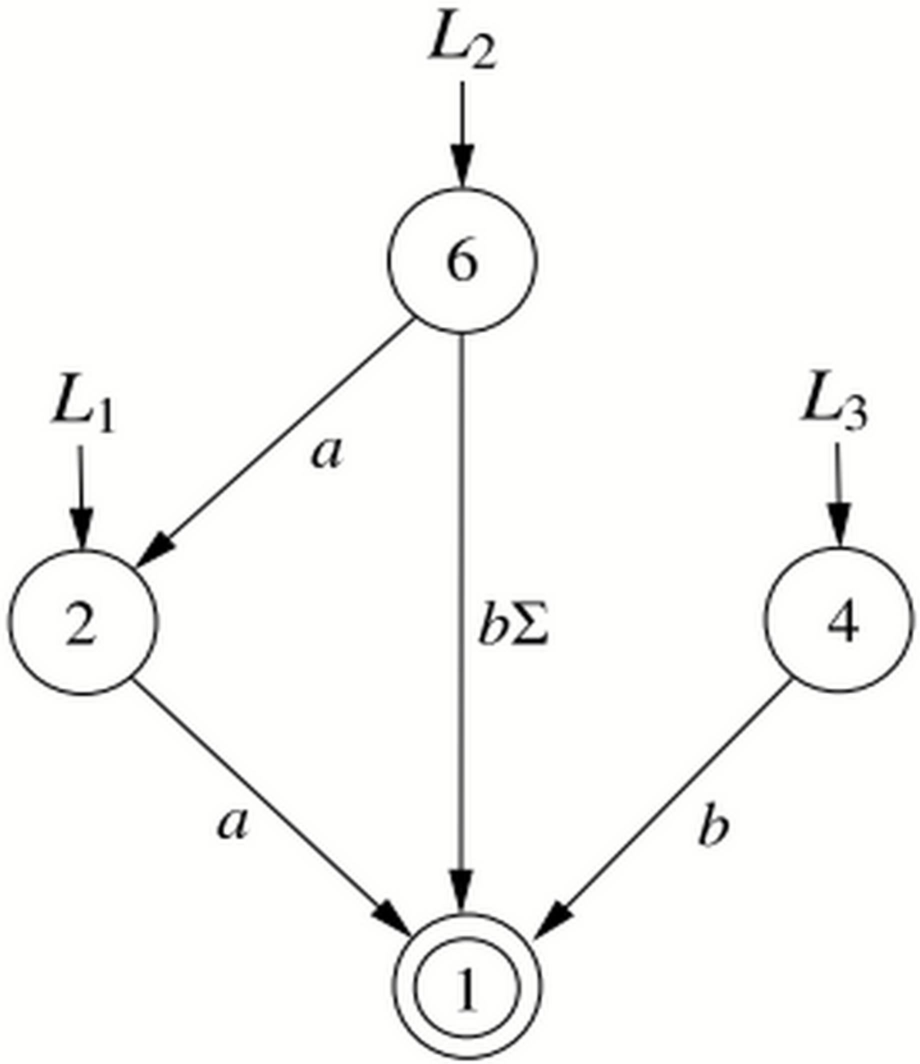




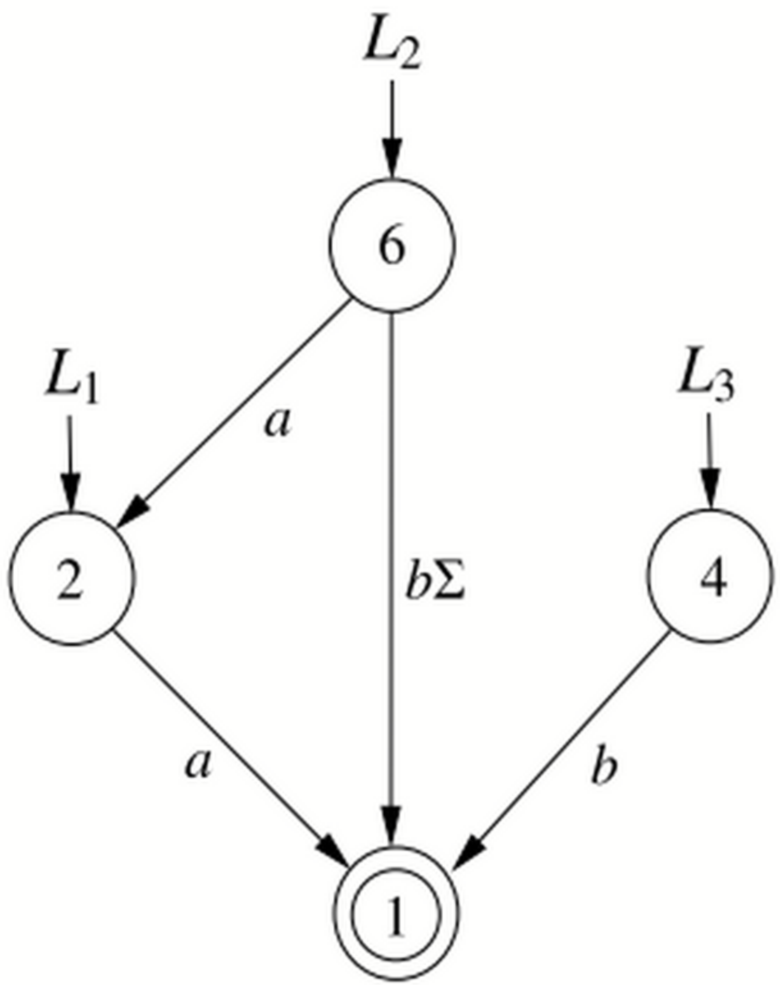
$\{aaa, aab, aba, baa, bab, bba, bbb\}$
$\{aab, abb, baa, bab, bbb\}$
$a$
$b$
$b\Sigma^2$
$a\Sigma$
$\{aa, ab, ba\}$
$\{aa, ab, bb\}$
$a\Sigma$
$b$
$a\Sigma$
$b$
$\{a\}$
$\{b\}$
$b$
$a$
$b$
$a$
$\{\epsilon\}$
$\emptyset$
$a, b$
$a, b$

Length: 2

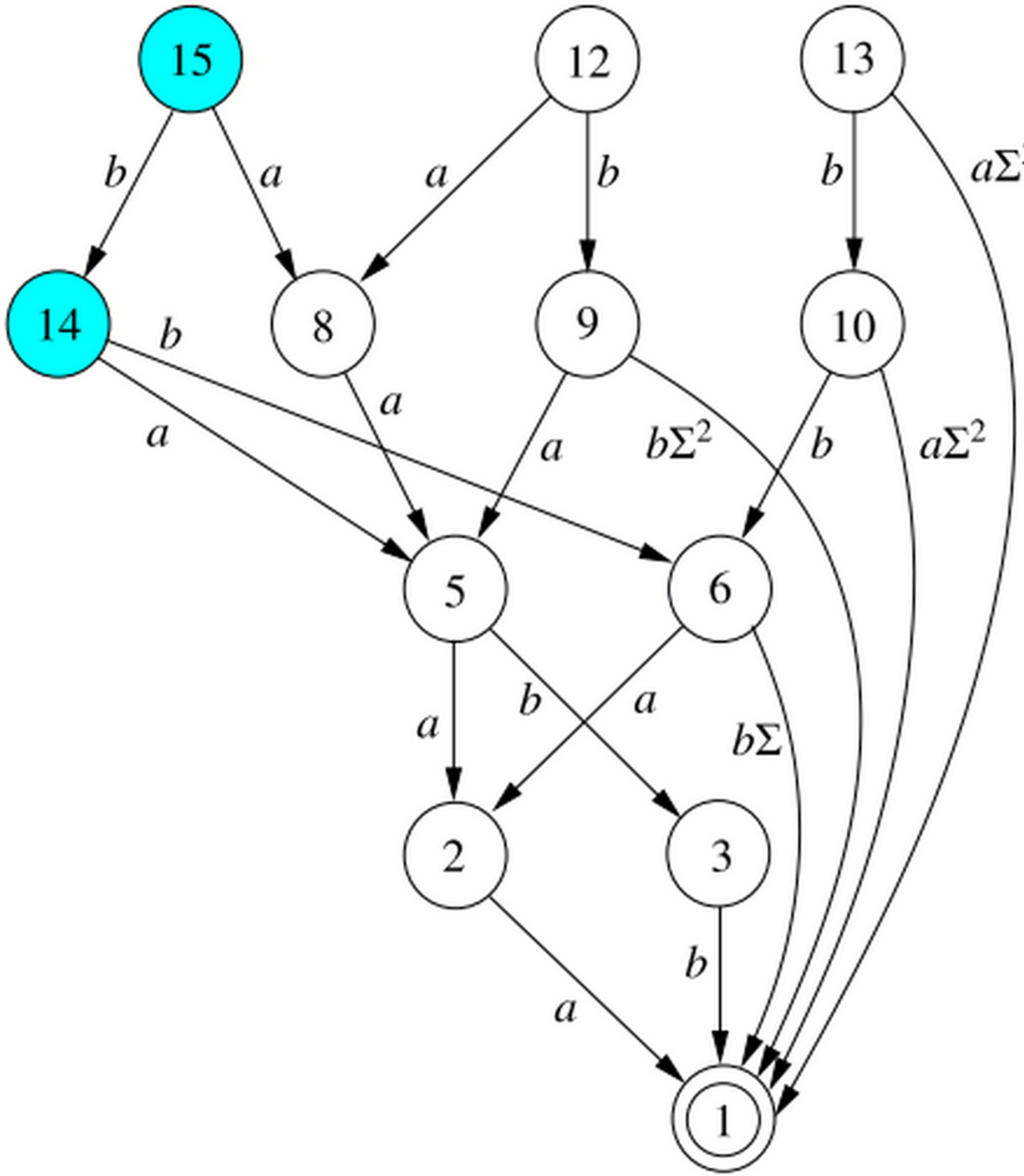


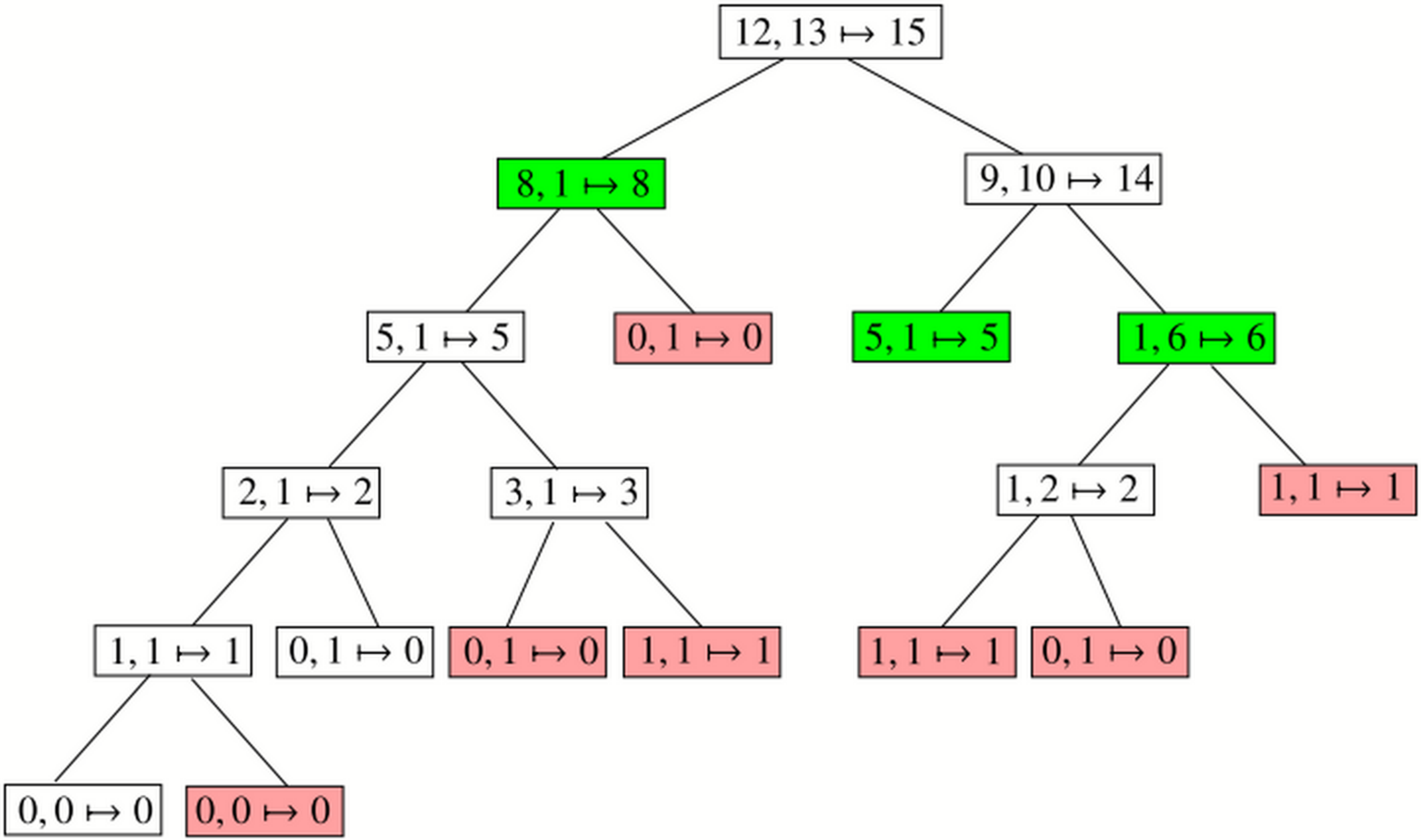



# Data structure for z-automata




| Ident. | Length | $a$-succ | $b$-succ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 |
| 4 | 1 | 0 | 1 |
| 6 | 2 | 2 | 1 |


15
12
13
b
a
a
b
b
aΣ³
14
8
9
10
b
a
a
bΣ²
b
aΣ²
5
6
a
b
a
bΣ
2
3
a
b
1


12, 13 ↦ 15
8, 1 ↦ 8
9, 10 ↦ 14
5, 1 ↦ 5
0, 1 ↦ 0
5, 1 ↦ 5
1, 6 ↦ 6
2, 1 ↦ 2
3, 1 ↦ 3
1, 2 ↦ 2
1, 1 ↦ 1
1, 1 ↦ 1
0, 1 ↦ 0
0, 1 ↦ 0
1, 1 ↦ 1
1, 1 ↦ 1
0, 1 ↦ 0
0, 0 ↦ 0
0, 0 ↦ 0