Automata theory

An algorithmic approach

Member(x, X)	:	returns **true** if $x \in X$, **false** otherwise.
Complement(X)	:	returns $U \setminus X$.
Intersection(X, Y)	:	returns $X \cap Y$.
Union(X, Y)	:	returns $X \cup Y$.
Empty(X)	:	returns **true** if $X = \emptyset$, **false** otherwise.
Universal(X)	:	returns **true** if $X = U$, **false** otherwise.
Included(X, Y)	:	returns **true** if $X \subseteq Y$, **false** otherwise.
Equal(X, Y)	:	returns **true** if $X = Y$, **false** otherwise.
Projection_1(R)	:	returns the set $\pi_1(R) = \{x \mid \exists y\ (x, y) \in R\}$.
Projection_2(R)	:	returns the set $\pi_2(R) = \{y \mid \exists y\ (x, y) \in R\}$.
Join(R, S)	:	returns $R \circ S\ = \{(x, z) \mid \exists y \in X\ (x, y) \in R \wedge (y, z) \in S\}$
Post(X, R)	:	returns $post_R(X) = \{y \in U \mid \exists x \in X : (x, y) \in R\}$.
Pre(X, R)	:	returns $pre_R(X) = \{y \in U \mid \exists x \in X : (y, x) \in R\}$.

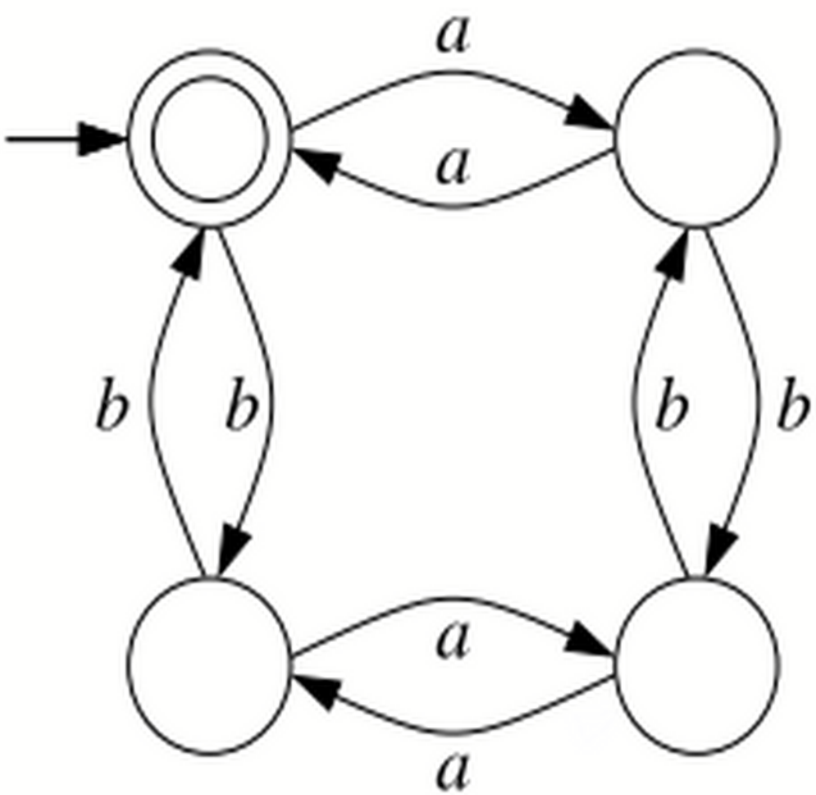
Automata classes and conversions

Regular expressions

$$r ::= \emptyset \mid \varepsilon \mid a \mid r_1 r_2 \mid r_1 + r_2 \mid r^* \qquad \textit{where } a \in \Sigma$$

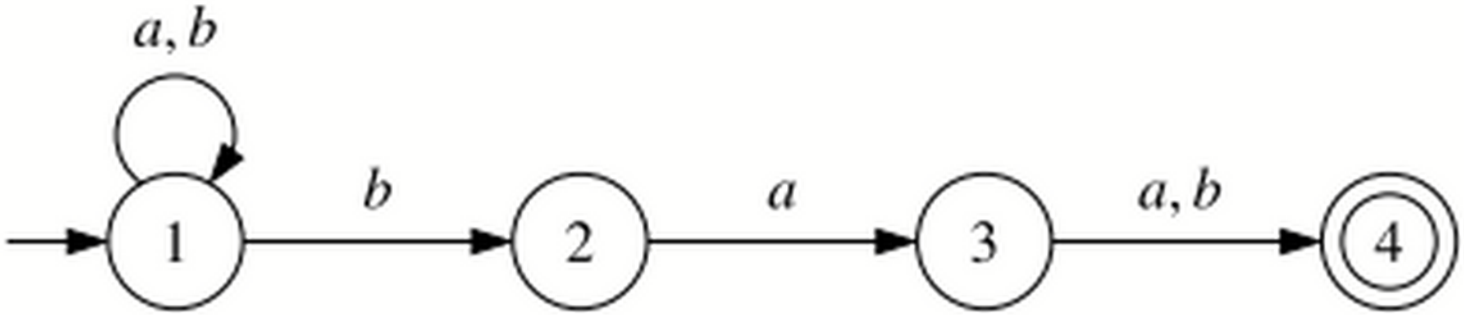
- $L(\emptyset) = \emptyset$,
- $L(\varepsilon) = \{\varepsilon\}$,
- $L(a) = \{a\}$,
- $L(r_1 r_2) = L(r_1) \cdot L(r_2)$, $\quad L_1 \cdot L_2 = \{w_1 w_2 \in \Sigma^* \mid w_1 \in L_1, w_2 \in L_2\}$.
- $L(r_1 + r_2) = L(r_1) \cup L(r_2)$,
- $L(r^*) = L(r)^*$. $\quad L^* = \bigcup_{i \geq 0} L^i$, where $L_0 = \{\varepsilon\}$ and $L_{i+1} = L^i \cdot L$

Deterministic finite automata (DFA)



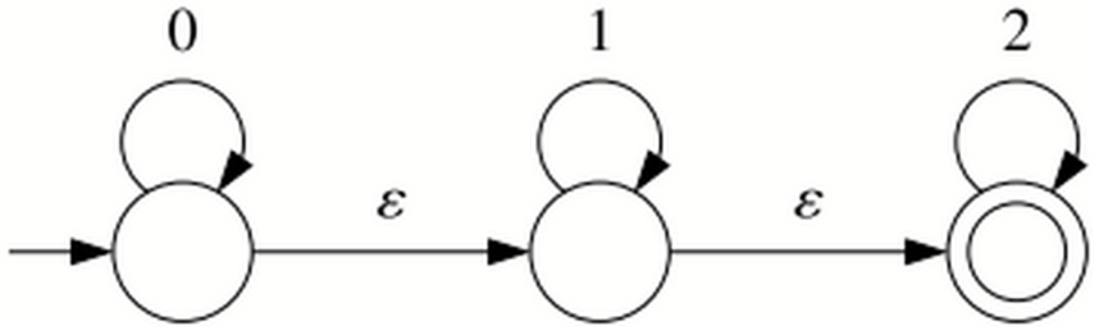
- Q *is a set of* states,
- Σ *is an alphabet,*
- $\delta\colon Q \times \Sigma \to Q$ *is a* transition function,
- $q_0 \in Q$ *is the* initial state, *and*
- $F \subseteq Q$ *is the set of* final states.

Nondeterministic finite automata (NFA)



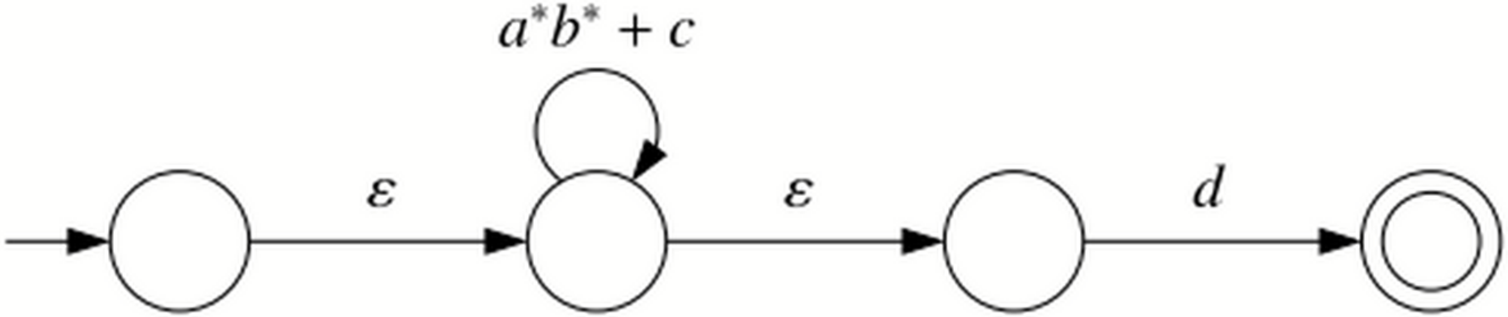
- $\delta\colon Q \times \Sigma \to \mathcal{P}(Q)$ *is a* transition relation.

Nondeterministic finite automata with epsilon-transitions (NFA-e)



- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$ *is a* transition relation.

Nondeterministic finite automata with regular-expression transitions (NFA-reg)



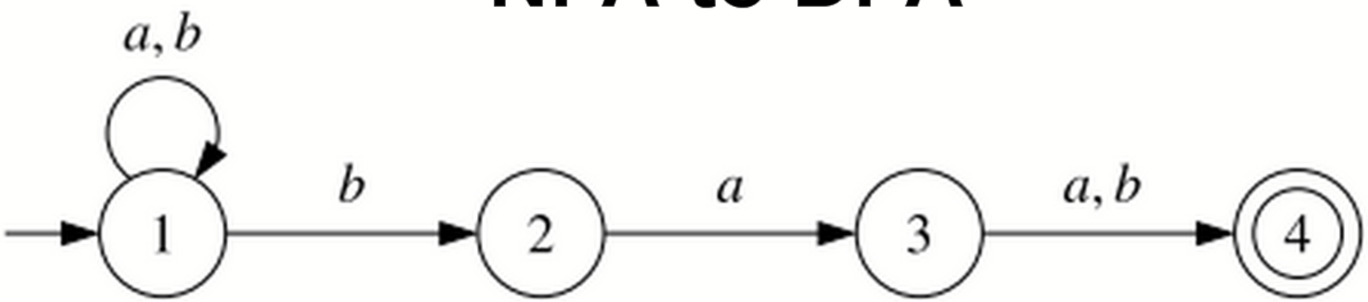
- $\delta\colon Q \times \mathcal{RE}(\Sigma) \to \mathcal{P}(Q)$ *is a relation such that* $\delta(q, r) = \emptyset$ *for all but a finite number of pairs* $(q, r) \in Q \times \mathcal{RE}(\Sigma)$.

Normal form

Definition 2.5 *Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton. A state $q \in Q$ is* reachable from $q' \in Q$ *if $q = q'$ or if there exists a run $q' \xrightarrow{a_1} \ldots \xrightarrow{a_n} q$ on some input $a_1 \ldots a_n \in \Sigma^*$. A is in* normal form *if every state is reachable from the initial state.*

Conversions

NFA to DFA



NFAtoDFA(A)

Input: NFA $A = (Q, \Sigma, \delta, q_0, F)$

Output: DFA $B = (\mathcal{Q}, \Sigma, \Delta, Q_0, \mathcal{F})$ with $L(B) = L(A)$

$\mathcal{Q}, \Delta, \mathcal{F} \leftarrow \emptyset$; $Q_0 \leftarrow \{q_0\}$

$\mathcal{W} = \{Q_0\}$

while $\mathcal{W} \neq \emptyset$ **do**

  pick Q' **from** $\mathcal{W}$

  add Q' **to** $\mathcal{Q}$

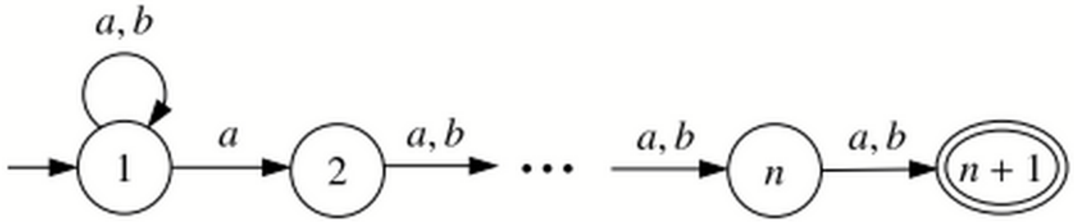
  if $Q' \cap F \neq \emptyset$ **then add** Q' **to** $\mathcal{F}$

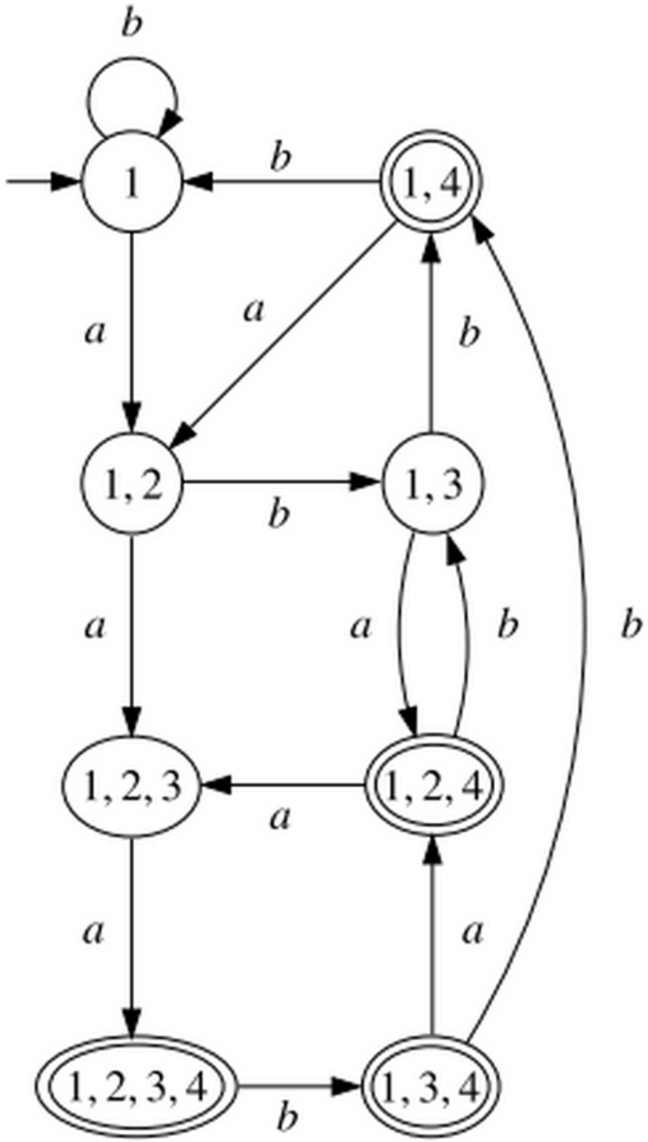
  for all $a \in \Sigma$ **do**

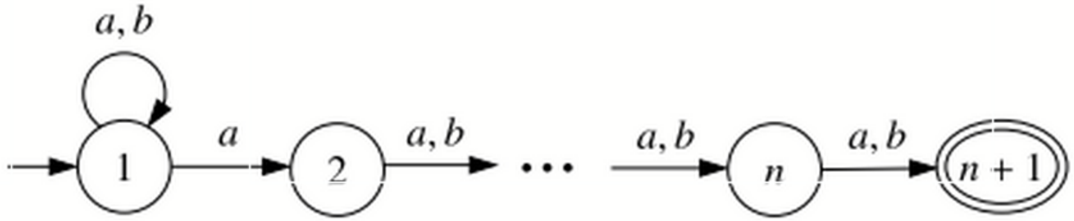
    $Q'' \leftarrow \bigcup_{q \in Q'} \delta(q, a)$

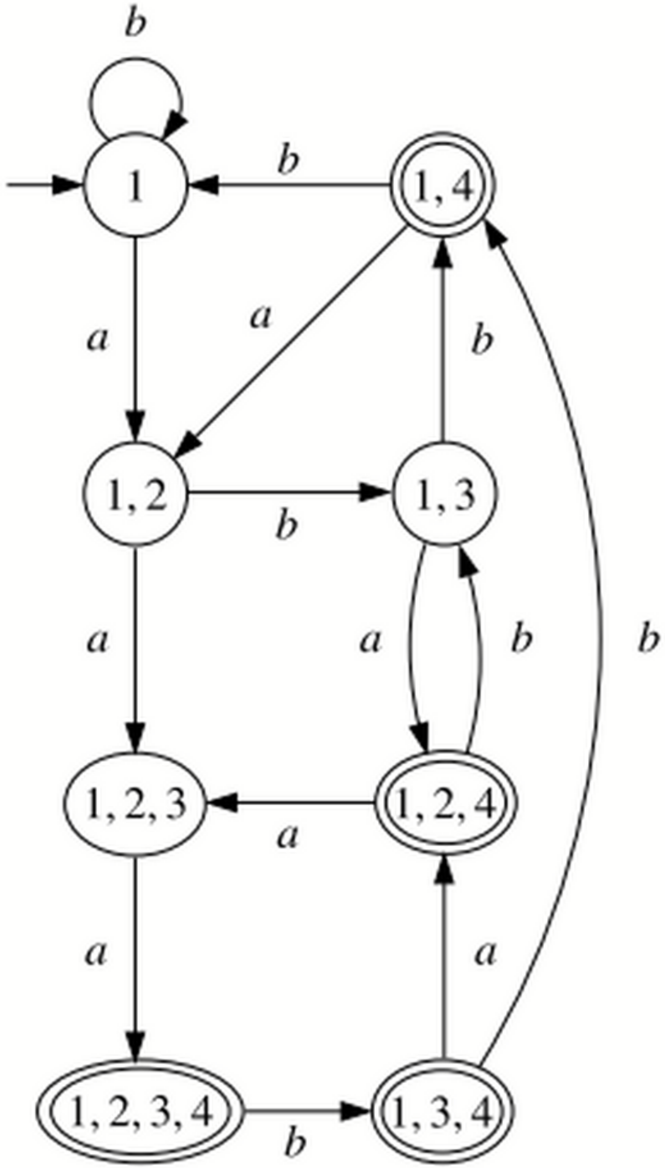
    if $Q'' \notin \mathcal{Q}$ **then add** Q'' **to** $\mathcal{W}$

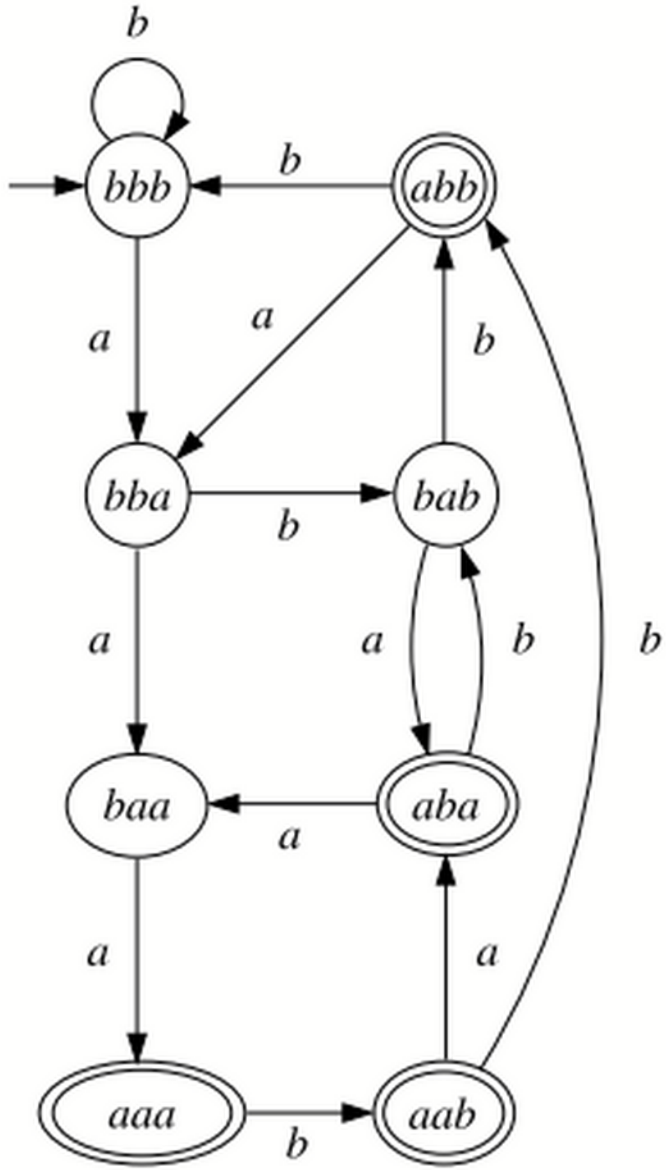
    add (Q', a, Q'') **to** Δ


a, b
1
a
2
a, b
...
a, b
n
a, b
n + 1

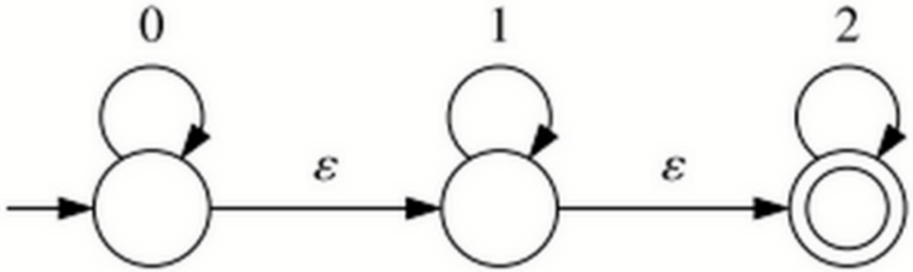

b
1
b
1, 4
a
a
b
1, 2
b
1, 3
a
a
b
b
1, 2, 3
a
1, 2, 4
a
a
1, 2, 3, 4
b
1, 3, 4


a, b
a
a, b
a, b
a, b
1
2
n
n + 1

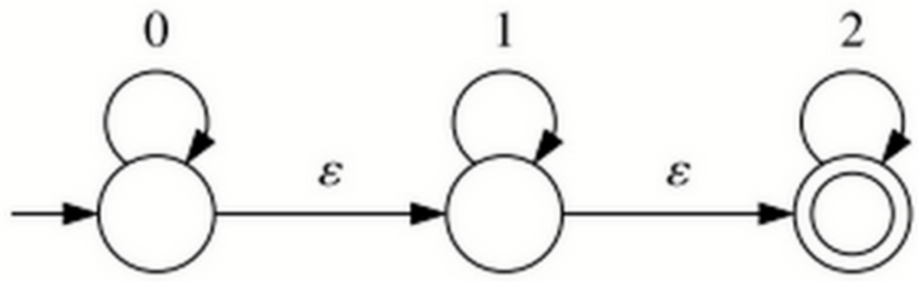

1
1, 2
1, 3
1, 4
1, 2, 3
1, 2, 4
1, 2, 3, 4
1, 3, 4

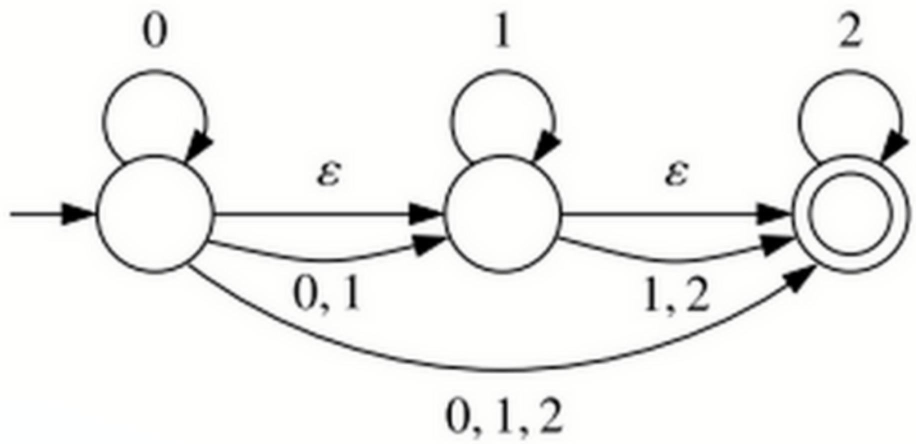

bbb
bba
bab
abb
baa
aba
aaa
aab

NFA-e to NFA

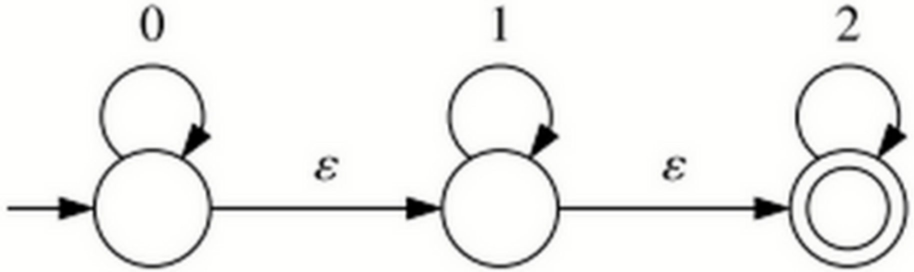


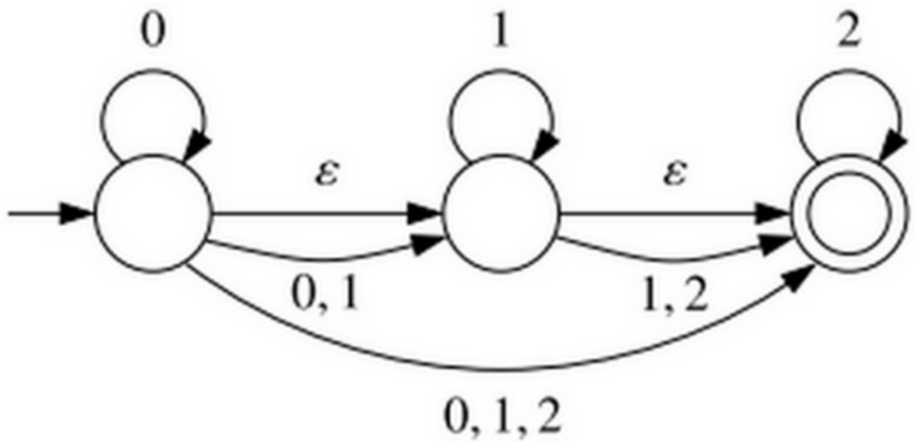
NFA-e to NFA

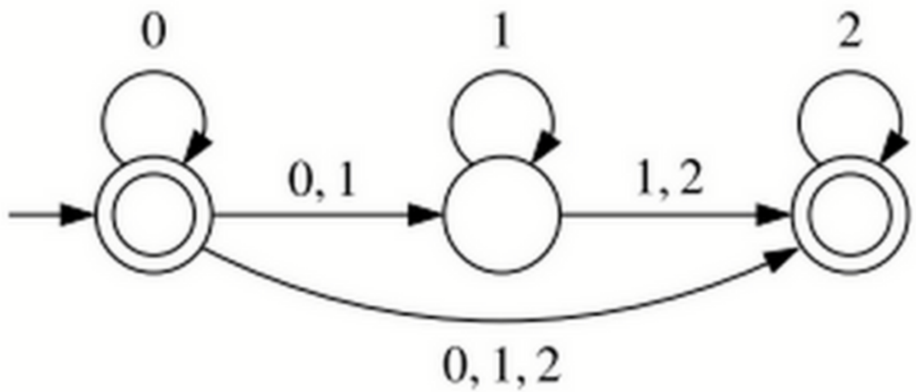




NFA-e to NFA







$\mathit{NFA\varepsilon toNFA}(A)$

Input: NFA-ε $A = (Q, \Sigma, \delta, q_0, F)$

Output: NFA $B = (Q', \Sigma, \delta', q_0', F')$ with $L(B) = L(A)$

$q_0' \leftarrow q_0$
$Q' \leftarrow \{q_0\}$; $\delta' \leftarrow \emptyset$; $F' \leftarrow F \cap \{q_0\}$
$\delta'' \leftarrow \emptyset$; $W \leftarrow \{(q, \alpha, q') \in \delta \mid q = q_0\}$
while $W \neq \emptyset$ **do**
    pick (q_1, α, q_2) **from** W
    if $\alpha \neq \varepsilon$ **then**
        add q_2 **to** Q'; **add** (q_1, α, q_2) **to** δ'; **if** $q_2 \in F$ **then add** q_2 **to** F'
        for all $q_3 \in \delta(q_2, \varepsilon)$ **do**
            if $(q_1, \alpha, q_3) \notin \delta'$ **then add** (q_1, α, q_3) **to** W
        for all $a \in \Sigma, q_3 \in \delta(q_2, a)$ **do**
            if $(q_2, a, q_3) \notin \delta'$ **then add** (q_2, a, q_3) **to** W
    else $/* \alpha = \varepsilon */$
        add (q_1, α, q_2) **to** δ''; **if** $q_2 \in F$ **then add** q_0 **to** F'
        for all $\beta \in \Sigma \cup \{\varepsilon\}, q_3 \in \delta(q_2, \beta)$ **do**
            if $(q_1, \beta, q_3) \notin \delta' \cup \delta''$ **then add** (q_1, β, q_3) **to** W

Proposition 2.7 *Let A be a NFA-ε, and let $B = NFA\varepsilon toNFA(A)$. Then B is a NFA and $L(A) = L(B)$.*

Proof:
(1) Termination:
every transition that leaves W is never added to W again
each iteration of the while loop removes a transition from W

(2) B is NFA. Easy.

(3) L(B) is included in L(A). Easy (transitions of B are shortcuts).

(4) L(A) is included in L(B).

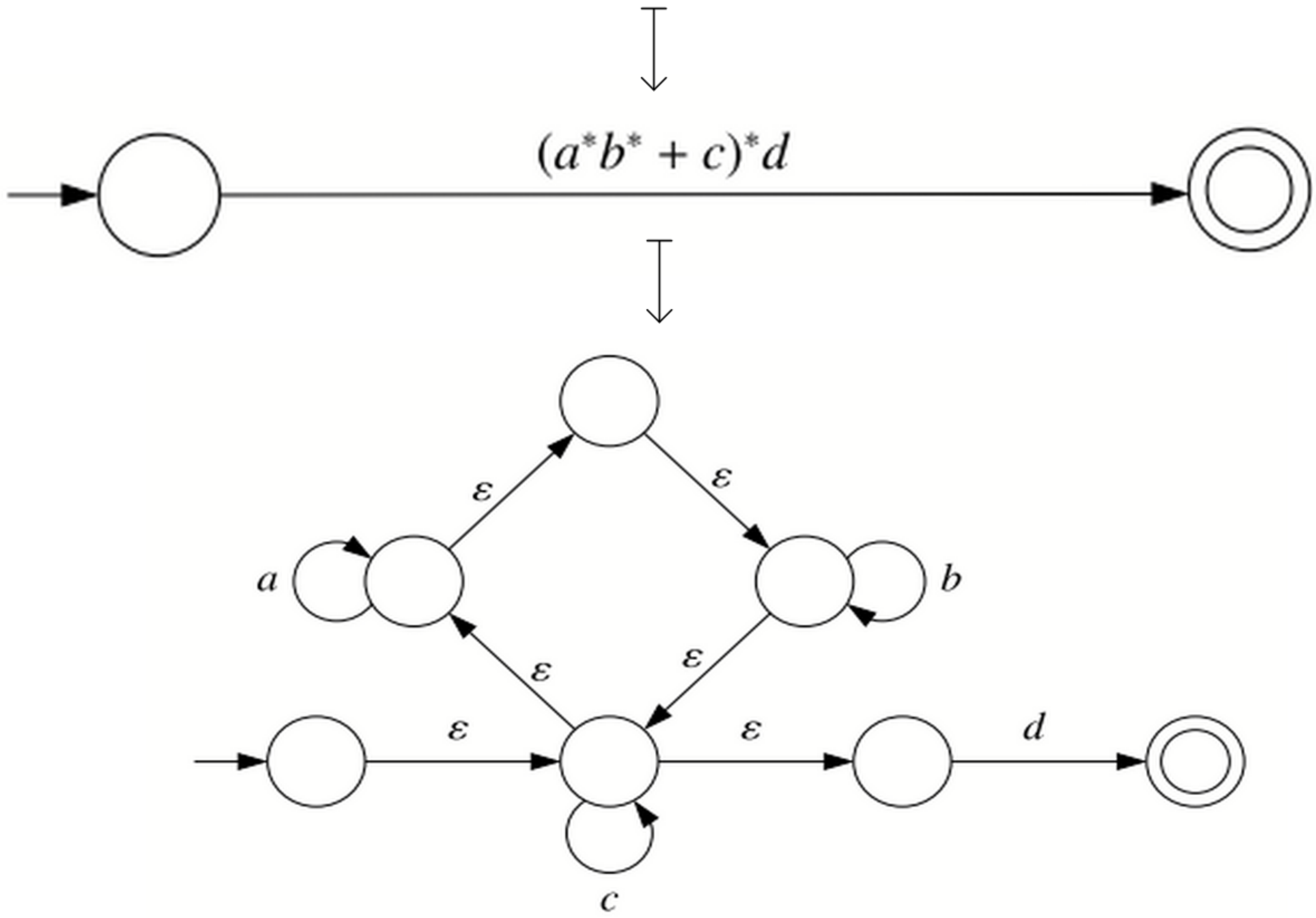
(4.1) If e belongs to L(A), then e belongs to L(B)

(4.2) If a nonempty word belongs to L(A), then it also belongs to L(B)

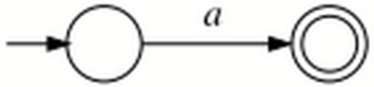
$$q_0 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} q_{m_1} \xrightarrow{a_1} q_{m_1+1} \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} q_{m_n} \xrightarrow{a_n} q_{m_n+1} \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} q_m$$

Regular expressions to NFA-e

$$(a^*b^* + c)^*d$$



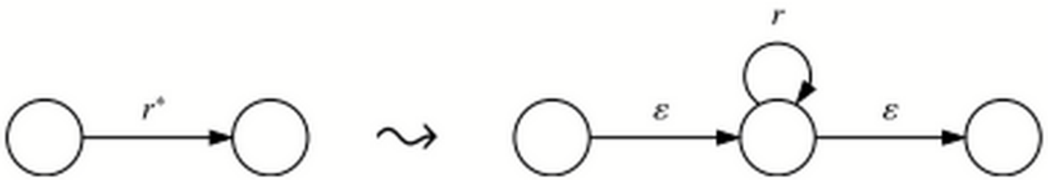
$$
\begin{array}{rclcrcl}
\emptyset \cdot r & \rightsquigarrow & \emptyset & \qquad & r \cdot \emptyset & \rightsquigarrow & \emptyset \\
r + \emptyset & \rightsquigarrow & r & & \emptyset + r & \rightsquigarrow & r \\
\emptyset^* & \rightsquigarrow & \epsilon & & & &
\end{array}
$$



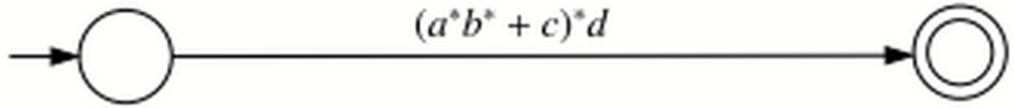
Automaton for the regular expression a, where $a \in \Sigma \cup \{\epsilon\}$

Rule for concatenation

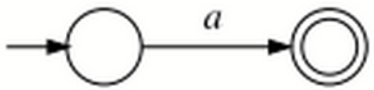
Rule for choice



Rule for Kleene iteration



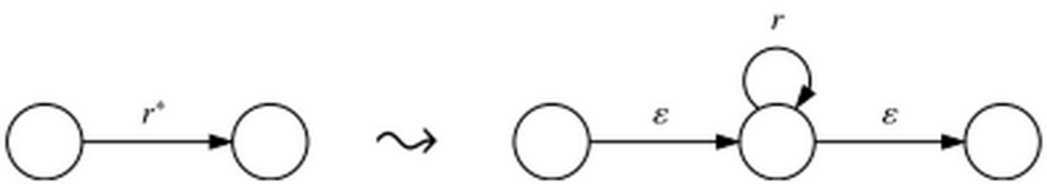
$$
\begin{array}{rcl@{\qquad}rcl}
\emptyset \cdot r & \leadsto & \emptyset & r \cdot \emptyset & \leadsto & \emptyset \\
r + \emptyset & \leadsto & r & \emptyset + r & \leadsto & r \\
\emptyset^* & \leadsto & \epsilon & & &
\end{array}
$$



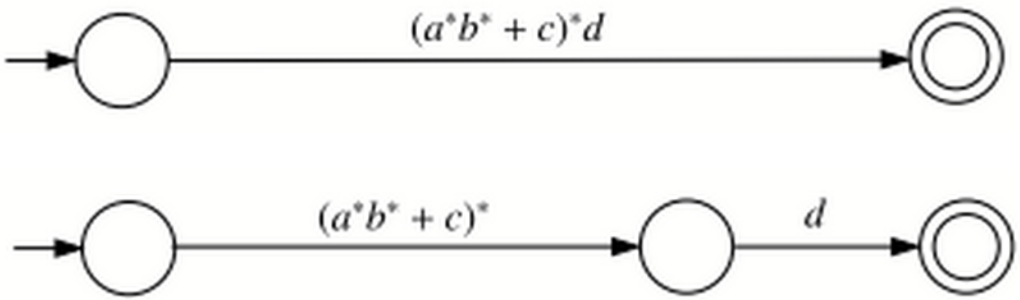
Automaton for the regular expression a, where $a \in \Sigma \cup \{\epsilon\}$

Rule for concatenation

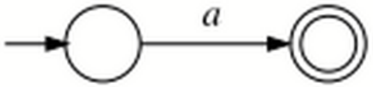
Rule for choice



Rule for Kleene iteration



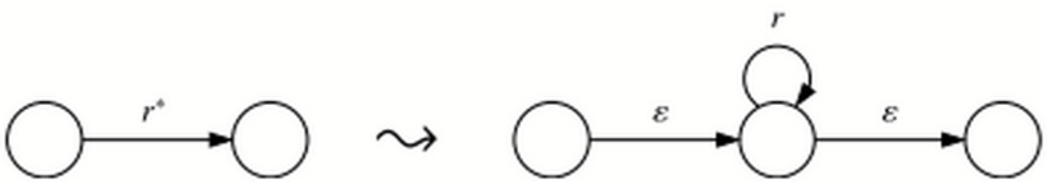
$$
\begin{array}{rcl@{\qquad}rcl}
\emptyset \cdot r & \rightsquigarrow & \emptyset & r \cdot \emptyset & \rightsquigarrow & \emptyset \\
r + \emptyset & \rightsquigarrow & r & \emptyset + r & \rightsquigarrow & r \\
\emptyset^* & \rightsquigarrow & \epsilon & & &
\end{array}
$$



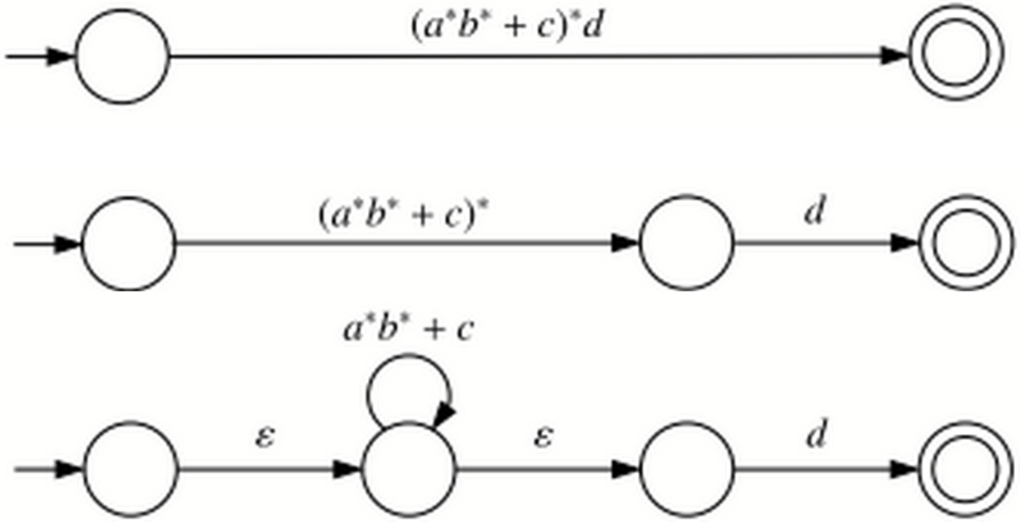
Automaton for the regular expression a, where $a \in \Sigma \cup \{\epsilon\}$

Rule for concatenation

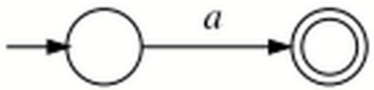
Rule for choice



Rule for Kleene iteration



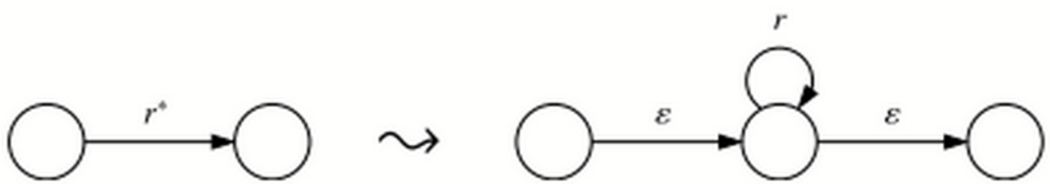
$$
\begin{array}{rclcrcl}
\emptyset \cdot r & \leadsto & \emptyset & \quad & r \cdot \emptyset & \leadsto & \emptyset \\
r + \emptyset & \leadsto & r & \quad & \emptyset + r & \leadsto & r \\
\emptyset^* & \leadsto & \epsilon & & & &
\end{array}
$$



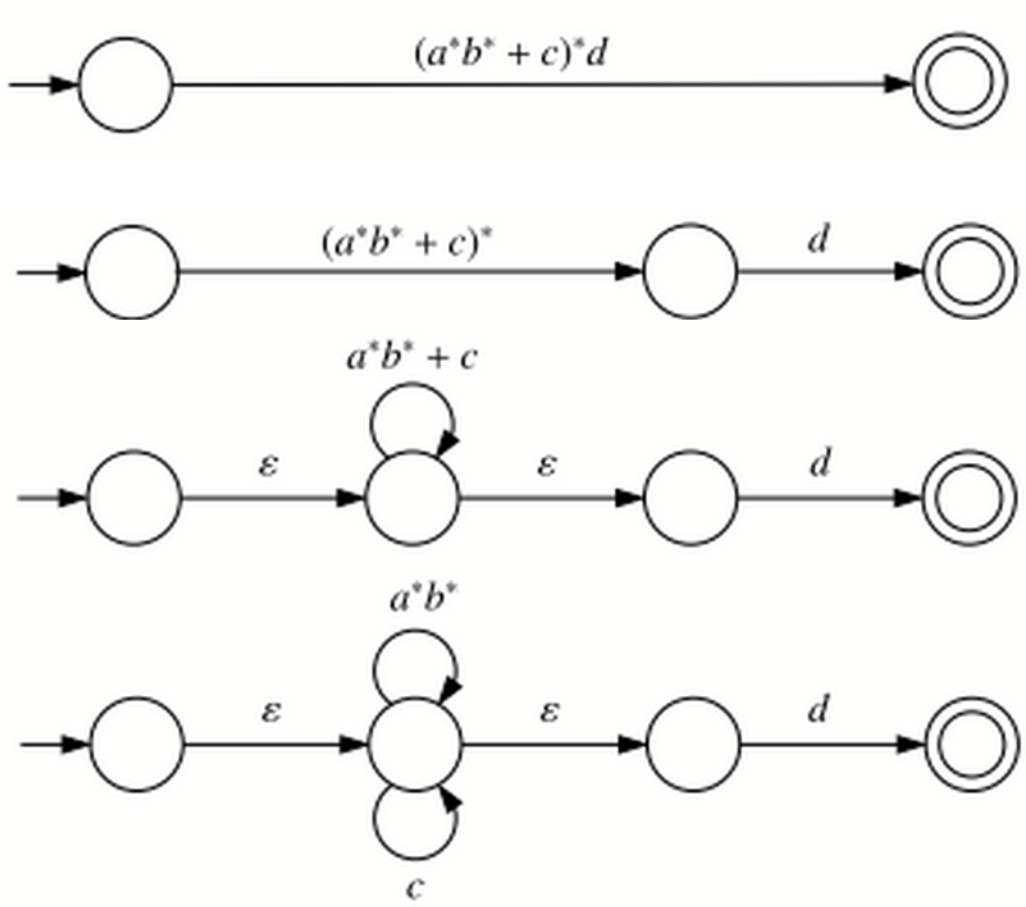
Automaton for the regular expression a, where $a \in \Sigma \cup \{\epsilon\}$

Rule for concatenation

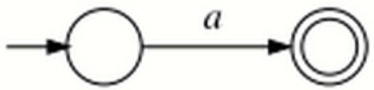
Rule for choice



Rule for Kleene iteration



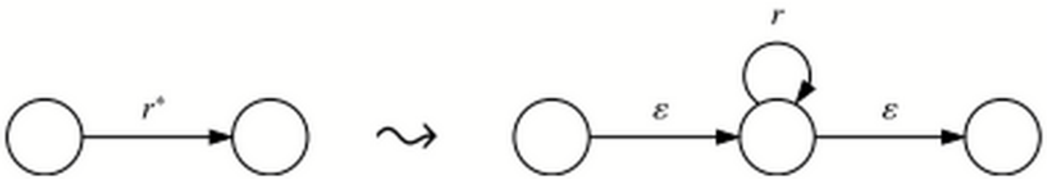
$$
\begin{array}{rcl@{\qquad}rcl}
\emptyset \cdot r & \rightsquigarrow & \emptyset & r \cdot \emptyset & \rightsquigarrow & \emptyset \\
r + \emptyset & \rightsquigarrow & r & \emptyset + r & \rightsquigarrow & r \\
\emptyset^* & \rightsquigarrow & \epsilon & & &
\end{array}
$$



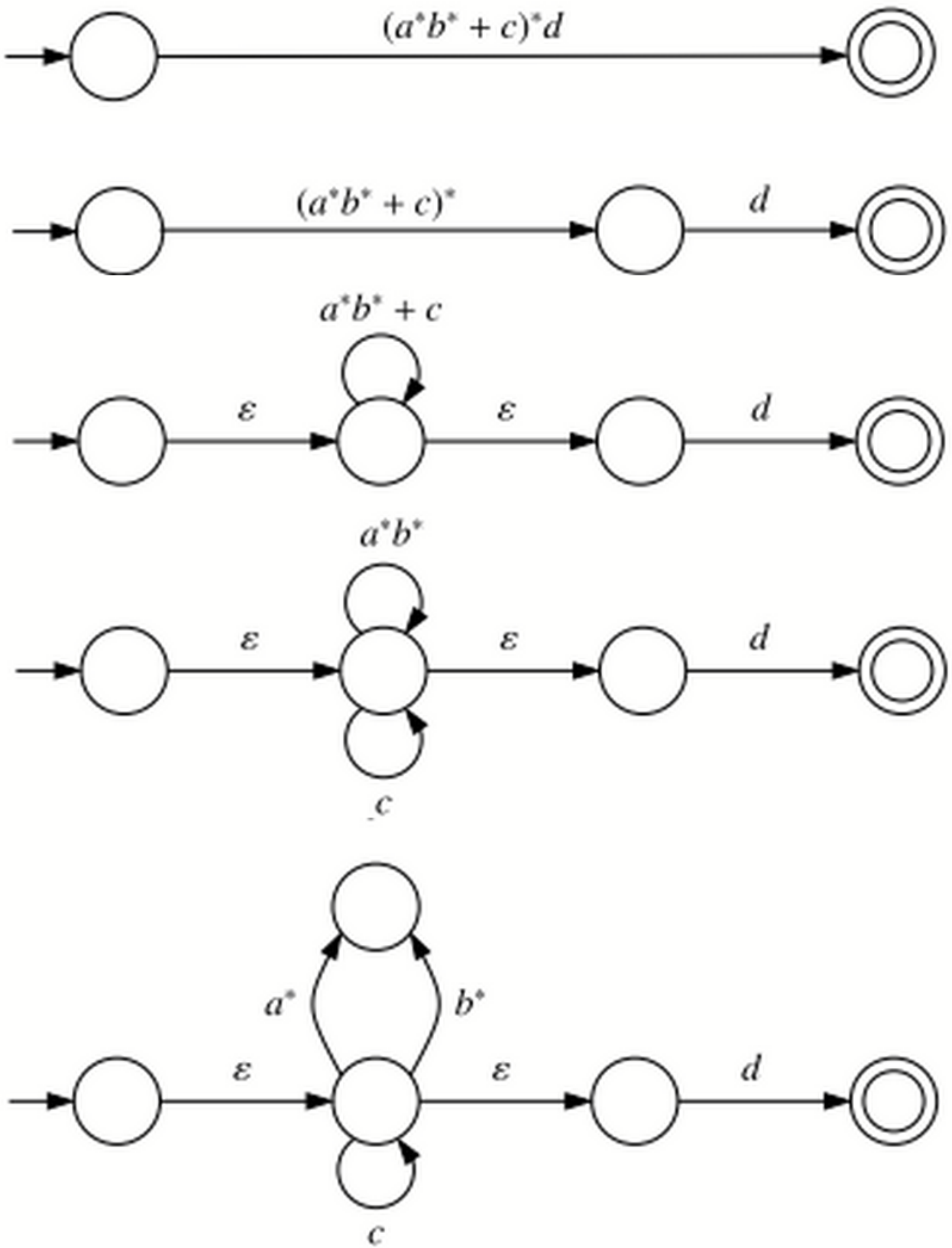
Automaton for the regular expression a, where $a \in \Sigma \cup \{\epsilon\}$

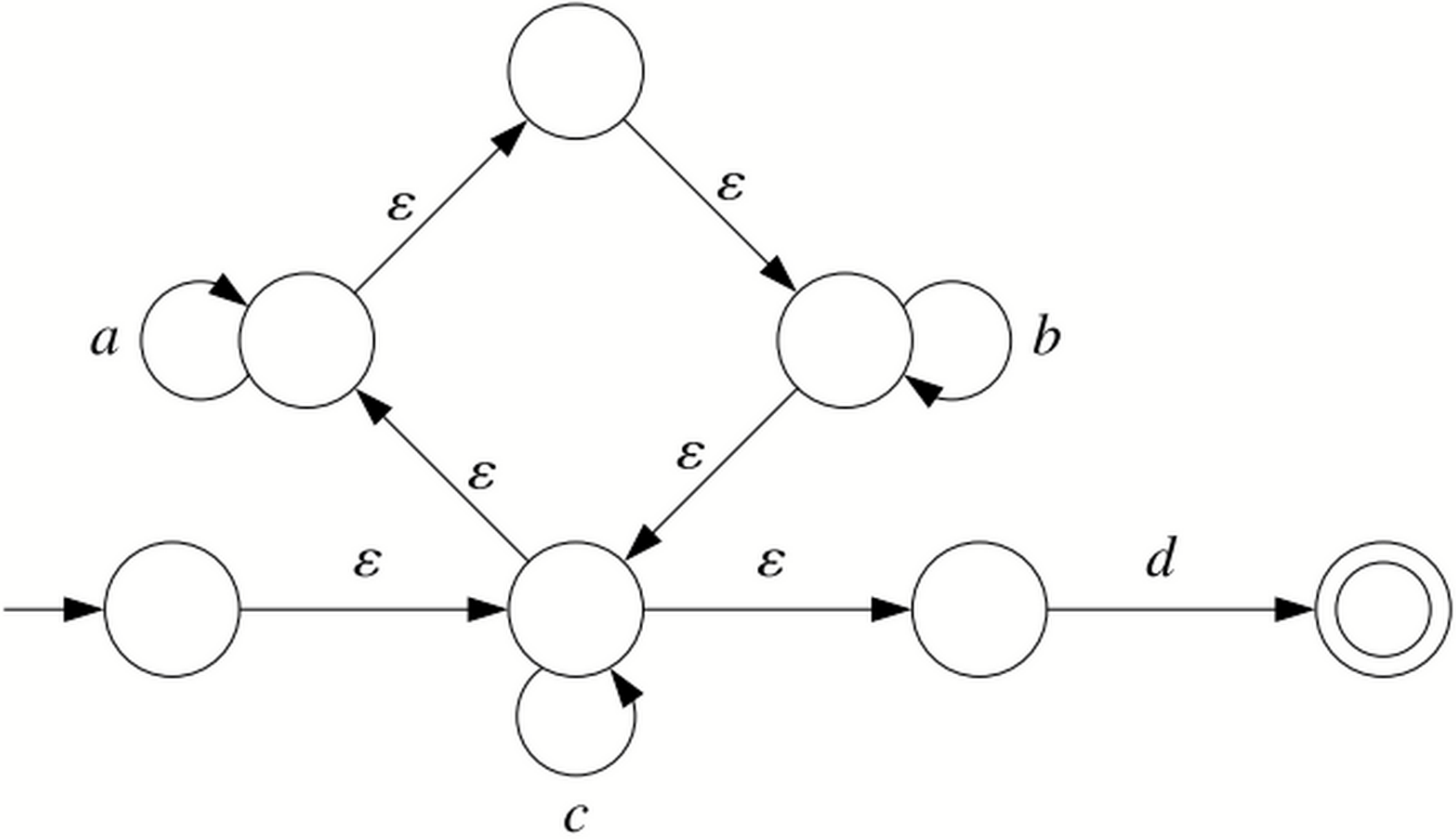
Rule for concatenation

Rule for choice



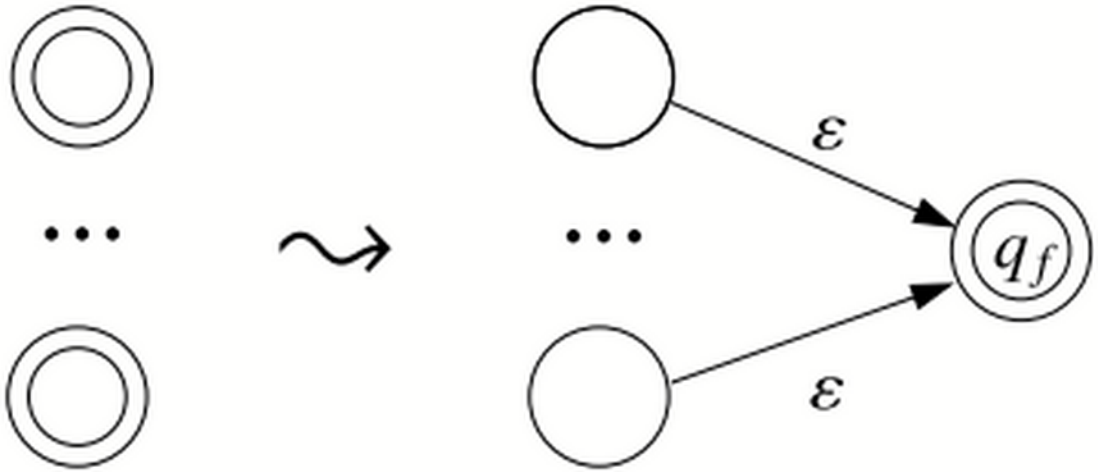
Rule for Kleene iteration



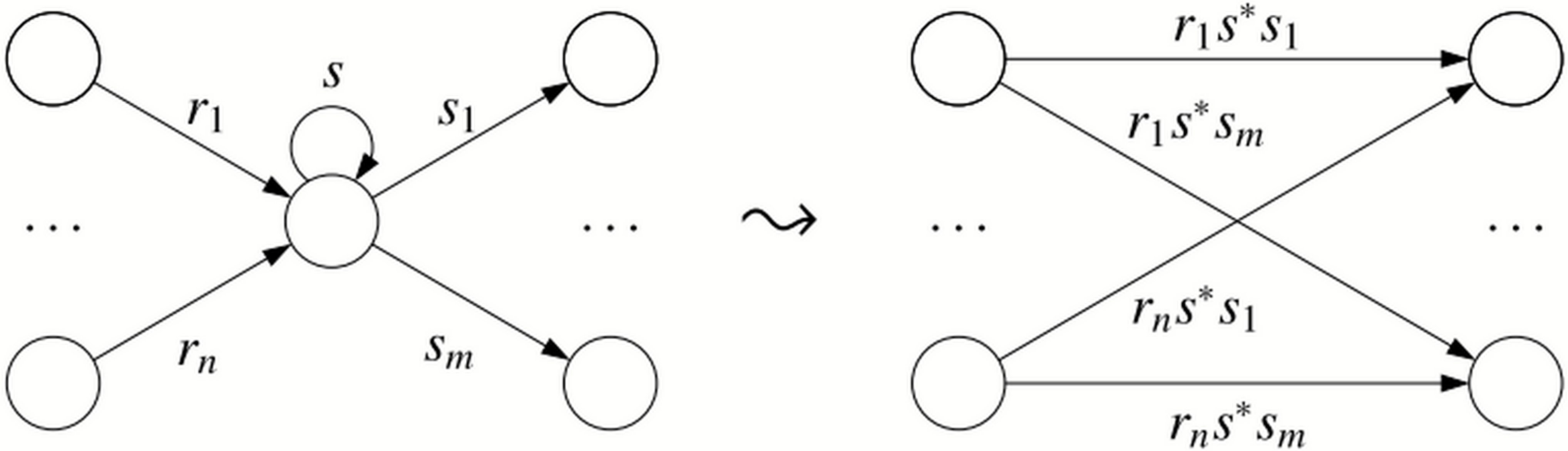

ε
ε
a
b
ε
ε
ε
ε
d
c

NFA-e to regular expressions

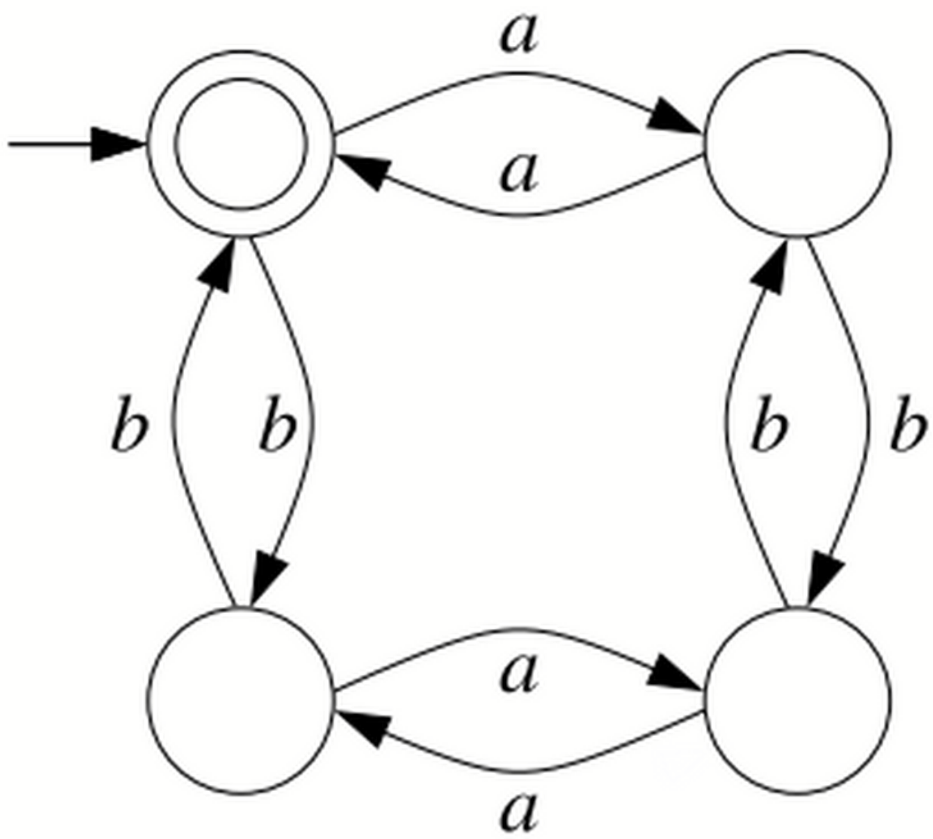
Preprocessing:

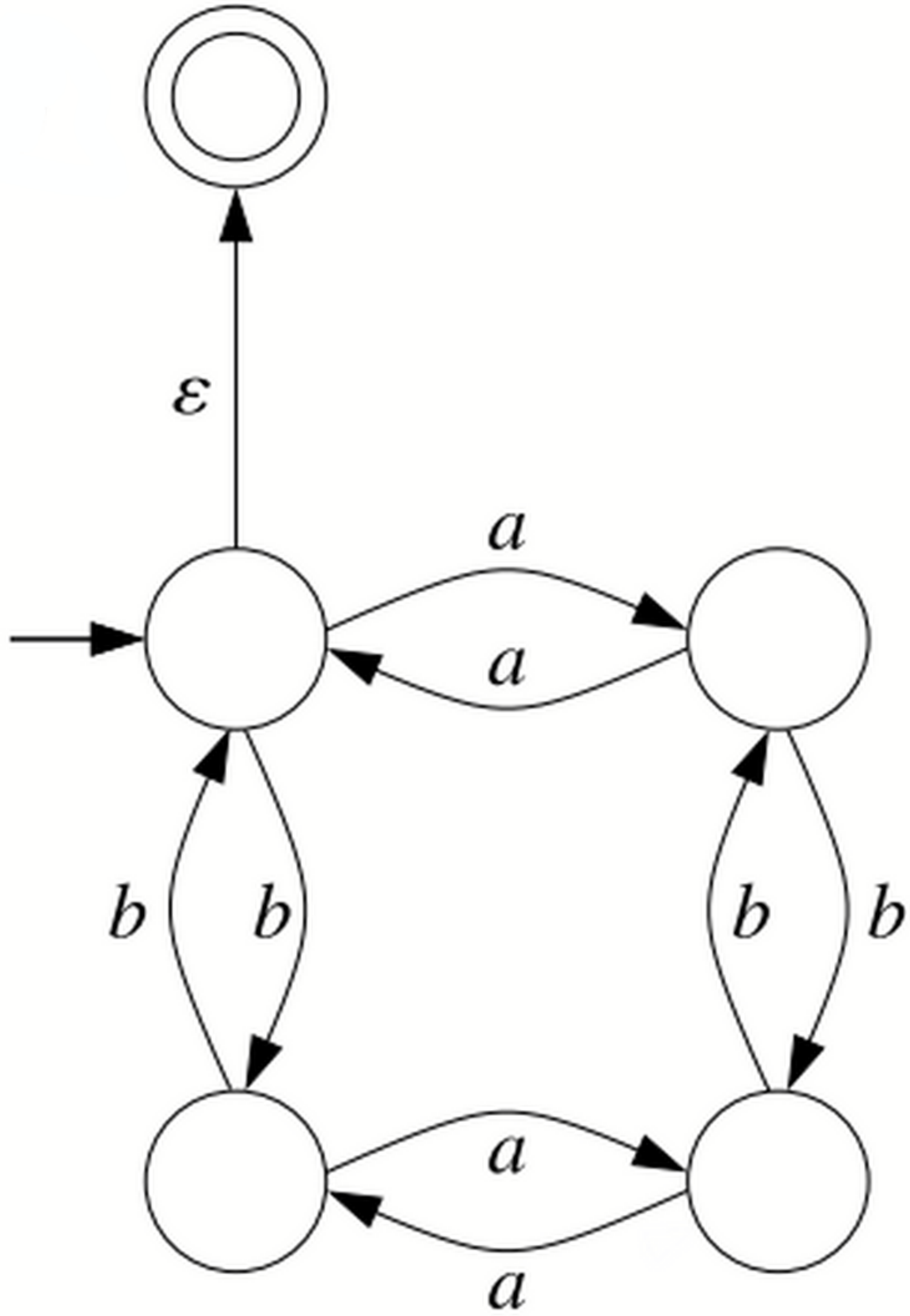


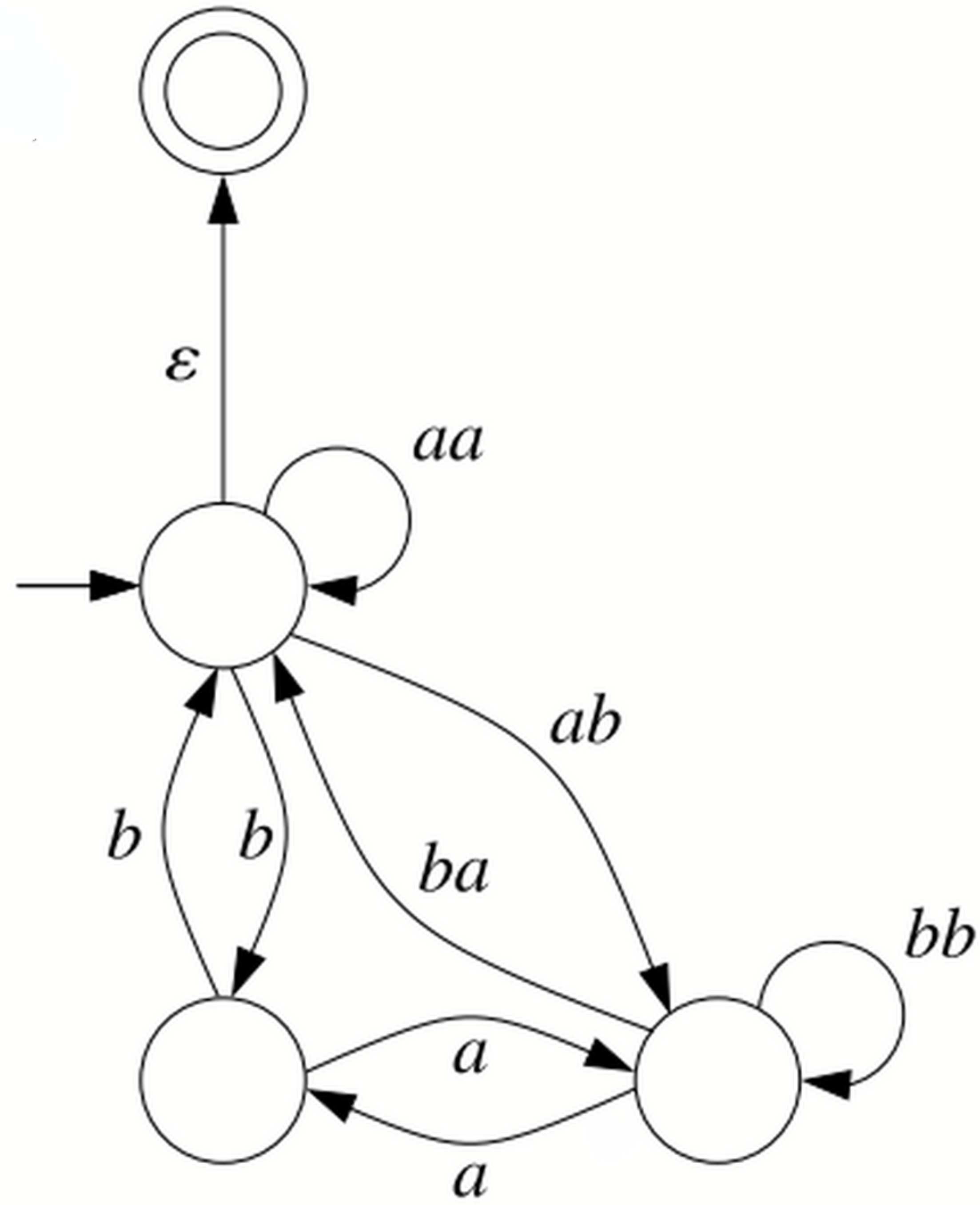
Processing:



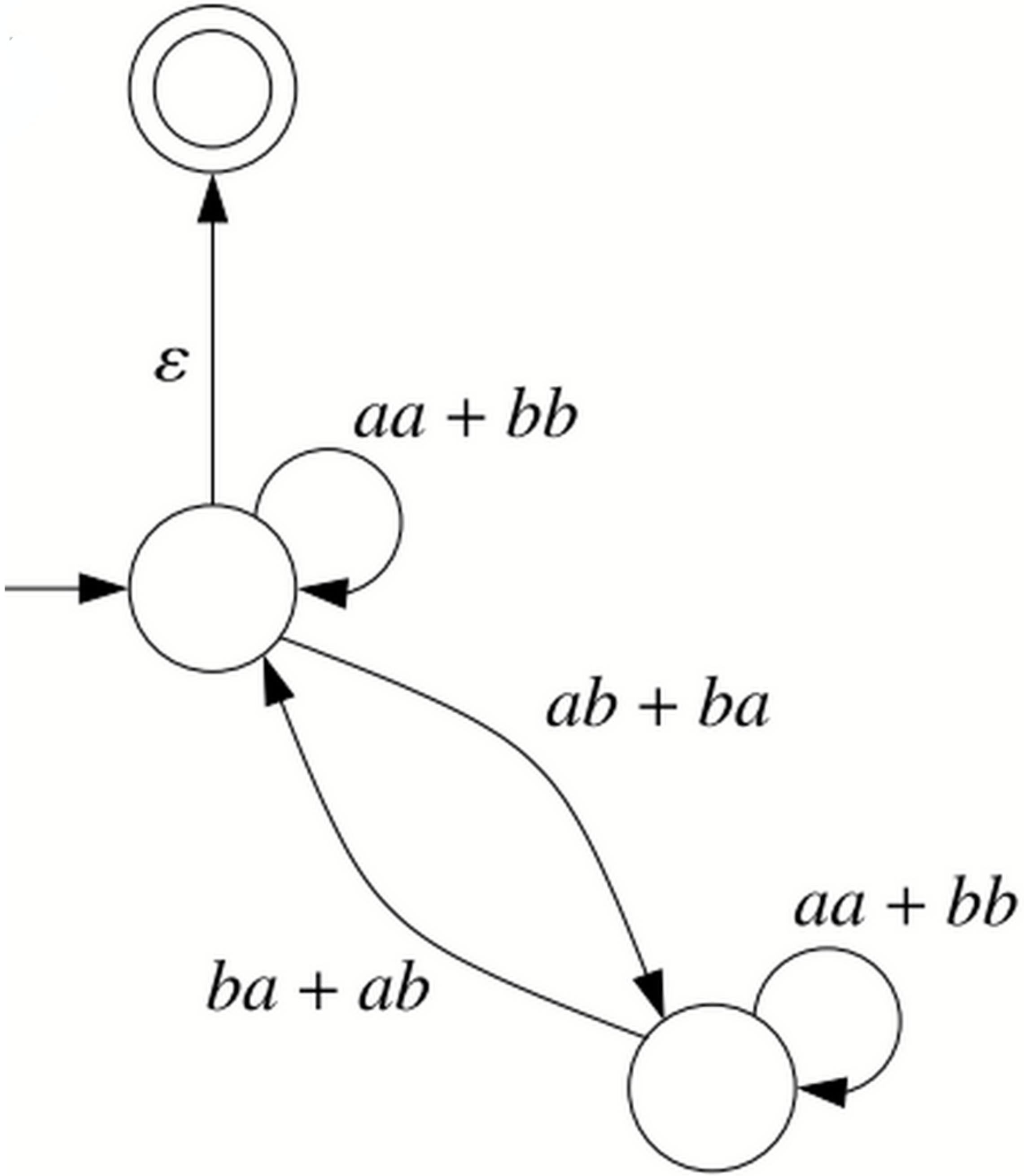
Postprocessing (if necessary):


a
a
b
b
b
b
a
a

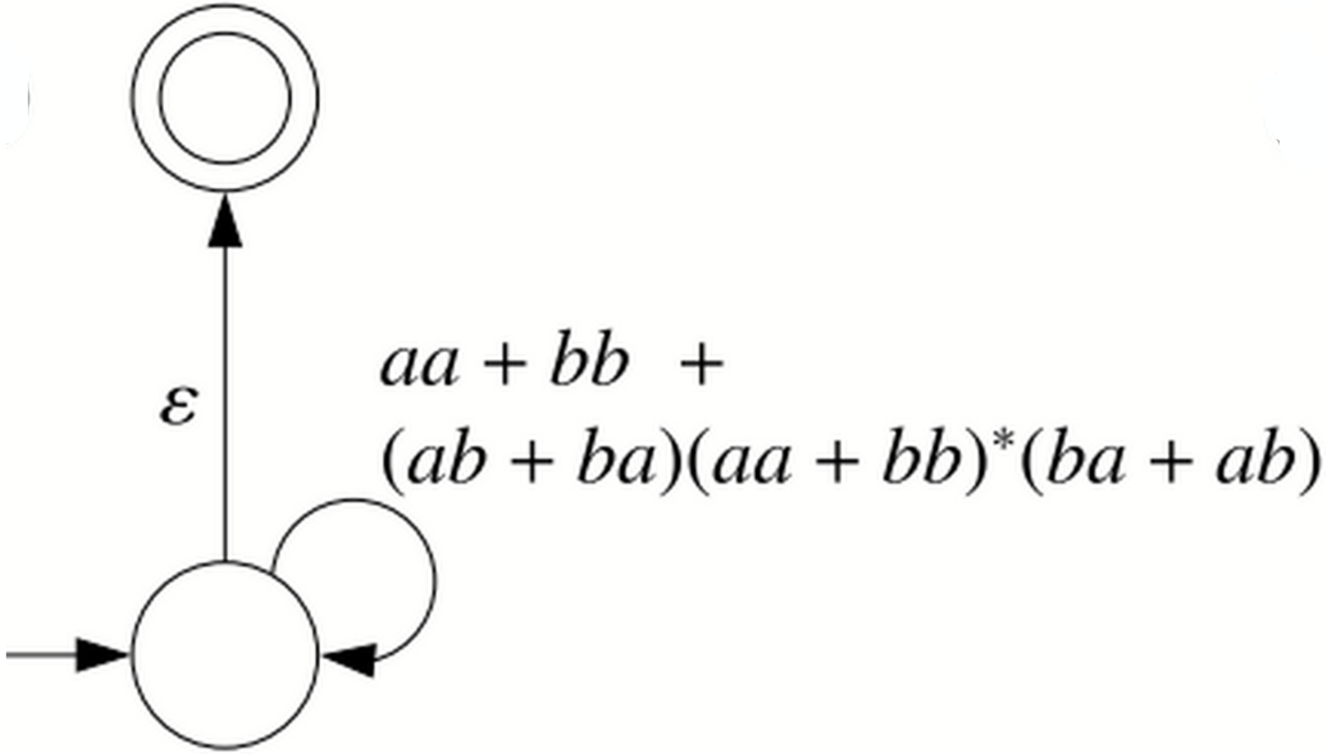

ε
a
a
b
b
b
b
a
a



ε
aa
ab
b
b
ba
bb
a
a



ε
aa + bb
ab + ba
aa + bb
ba + ab


ε
$aa + bb\ \ +$
$(ab + ba)(aa + bb)^*(ba + ab)$