Logics on words

Regular expressions give operational descriptions of regular languages.

Often the natural description of a language is denotational:

- even number of a's and even number of b's
  (aa+bb+(ab+ba)(aa+bb)*(ba+ab))*
- words not containing "hello"

Goal: find a denotational language able to express all the regular languages, and only the regular languages.

Logics on words

Idea: use a logic that has an interpretation on words

A formula expresses a property that each word may satisfy or not, like

" the word contains only a's "
" the word has even length "
" between every occurrence of an 'a' and a 'b' there is an occurrence of a 'c' "

Every formula (indirectly) defines a language: the language of all the words (over the given fixed alphabet) that satisfy it.

Atomic formulas: for each letter a we introduce the formula Q_a(x), with intuitive meaning: the letter at position x (starting at 0) is an "a".

Formulas:

Definition 7.1 *Let* $V = \{x, y, z, \ldots\}$ *be an infinite sets of* variables, *and let* $\Sigma = \{a, b, c, \ldots\}$ *be a finite alphabet. The set* $FO(\Sigma)$ *of* first-order formulas *over* Σ *is the set of expressions generated by the grammar:*

$$\varphi := Q_a(x) \mid x < y \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \exists x\varphi$$

where $a \in \Sigma$.

Observe: the logic is parametrized by the alphabet.

Abbreviations

- $\forall x\varphi := \neg\exists x\neg\varphi$
- $\varphi_1 \wedge \varphi_2 := \neg(\neg\varphi_1 \vee \neg\varphi_2)$
- $\varphi_1 \rightarrow \varphi_2 := \neg\varphi_1 \vee \varphi_2$
- $\text{zero}(x) := \neg\exists y\ y < x$ — "x is the first position (numbered by 0)"
- $\text{last}(x) := \neg\exists y\ x < y$ — "x is the last position"
- $y = x + 1 := x < y \wedge \neg\exists z(x < z \wedge z < y)$ — "y is the successor position of x"
- $y = x + 2 := \exists z(z = x + 1 \wedge y = z + 1)$
- $y = x + (k + 1) := \exists z(z = x + k \wedge y = z + 1)$

Examples (without formal semantics yet)

- "The last letter is a b and before it there are only a's."

$$\exists Q_b(x) \wedge \forall x\,(\text{last}(x) \rightarrow Q_b(x) \wedge \neg\text{last}(x) \rightarrow Q_a(x))$$

- "Every a is immediately followed by a b."

$$\forall x\,(Q_a(x) \rightarrow \exists y\,(y = x + 1 \wedge Q_b(y)))$$

- "Every a is immediately followed by a b, unless it is the last letter."

$$\forall x\,(Q_a(x) \rightarrow \forall y\,(y = x + 1 \rightarrow Q_b(y)))$$

- "Between every a and every later b there is a c."

$$\forall x \forall y\,(Q_a(x) \wedge Q_b(y) \wedge x < y \rightarrow \exists z\,(x < z \wedge z < y \wedge Q_c(z)))$$

Semantics

Definition 7.2 *An* interpretation *of a formula φ of $FO(\Sigma)$ is a pair $(w, \mathfrak{I})$ where*

- $w \in \Sigma^*$
- $\mathfrak{I}$ *is a mapping that assigns every free variable x a* position $\mathfrak{I}(x) \in \{0, \ldots, |w|-1\}$.

If the formula is a sentence, then $\mathfrak{I}$ is the empty mapping, and we identify $(w, \mathfrak{I})$ and w.

Definition 7.3 *The satisfaction relation $(w, \mathfrak{I}) \models \varphi$ between a formula φ of $FO(\Sigma)$ and an interpretation $(w, \mathfrak{I})$ of φ is defined by:*

$(w, \mathfrak{I})$	$\models$	$Q_a(x)$	**iff**	$w[\mathfrak{I}(x)] = a$
$(w, \mathfrak{I})$	$\models$	$x < y$	**iff**	$\mathfrak{I}(x) < \mathfrak{I}(y)$
$(w, \mathfrak{I})$	$\models$	$x \in X$	**iff**	$\mathfrak{I}(x) \in \mathfrak{I}(X)$
$(w, \mathfrak{I})$	$\models$	$\exists x\varphi$	**iff**	$\lvert w\rvert > 0$ *and some* $i \in \{0, \ldots, \lvert w\rvert - 1\}$ *satisfies* $w, \mathfrak{I}[i/x] \models \varphi$

where $w[i] = w_i$ if $w = w_0 \ldots w_{|w|-1}$, and where $\mathfrak{I}[i/x]$ is the interpretation that assigns i to x and otherwise coincides with $\mathfrak{I}$. If $(w, \mathfrak{I}) \models \varphi$ we say that $(w, \mathfrak{I})$ is a model *of φ. Two formulas are* equivalent *if they have the same models.*

... is as usual a pain

It satisfies all universally quantified formulas, and no existentially quantified formula.

Can we only express regular languages? Can we express all regular languages

We consider one-letter alphabets.

Proposition: a language over a one-letter alphabet is expressible in FO logic iff it is finite or co-finite.

Consequence: we can only express regular languages, but not all, not even the language of words of even length.

So we extend the logic ...

First-order variables: interpreted on positions

Monadic second-order variables: interpreted on sets of positions.

Diadic second-order variables: interpreted on relations over positions

Monadic third-order variables: interpreted on sets of sets

New atomic formulas: $x \in X$

Expressing that a word has even length

Idea: express "there is a set X of positions such that
- X contains exactly the even positions, and
- the last position belongs to X"

Definition 7.10 *Let* $X_1 = \{x, y, z, \ldots\}$ *and* $X_2 = \{X, Y, Z, \ldots\}$ *be two infinite sets of* first-order *and* second-order variables. *Let* $\Sigma = \{a, b, c, \ldots\}$ *be a finite alphabet. The set* $MSO(\Sigma)$ *of* monadic second-order formulas *over* Σ *is the set expressions generated by the grammar:*

$$\varphi := Q_a(x) \mid x < y \mid x \in X \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \exists x\varphi \mid \exists X\varphi$$

An interpretation *of a formula* φ *is a pair* $(w, \mathcal{I})$ *where* $w \in \Sigma^*$, *and* $\mathcal{I}$ *is a mapping that assigns every free first-order variable* x *a* position $\mathcal{I}(x) \in \{0, \ldots, |w| - 1\}$ *and every free second-order variable* X *a set of positions* $\mathcal{I}(X) \subseteq \{0, \ldots, |w| - 1\}$.

The satisfaction relation $(w, \mathfrak{I}) \models \varphi$ *between a formula* φ *of* $MSO(\Sigma)$ *and an interpretation* $(w, \mathfrak{I})$ *of* φ *is defined as for* $FO(\Sigma)$*, with the following additions:*

$$
\begin{array}{llll}
(w, \mathfrak{I}) & \models & x \in X & \textbf{iff} \quad \mathfrak{I}(x) \in \mathfrak{I}(X) \\
(w, \mathfrak{I}) & \models & \exists X \varphi & \textbf{iff} \quad |w| > 0 \text{ and some } S \subseteq \{0, \ldots, |w| - 1\} \text{ satisfies } w, \mathfrak{I}[S/X] \models \varphi
\end{array}
$$

where where $\mathfrak{I}[S/X]$ *is the interpretation that assigns* S *to* X *and otherwise coincides with* $\mathfrak{I}$*. If* $(w, \mathfrak{I}) \models \varphi$ *we say that* $(w, \mathfrak{I})$ *is a* model *of* φ*. Two formulas are* equivalent *if they have the same models. The language* $\mathcal{L}(\varphi)$ *of a sentence* $\varphi \in MSO(\Sigma)$ *is the set* $\mathcal{L}(\varphi) = \{w \in \Sigma^* | w \models \phi\}$*. A language* $L \subseteq \Sigma^*$ *is* MSO-definable *if* $L = \mathcal{L}(\varphi)$ *for some formula* $\varphi \in MSO(\Sigma)$*.*

There is a block of consecutive positions X such that: before X there are only c's; after X there are only d's; in X b's and a's alternate; the first letter in X is an a and the last is a b.

$$\exists X\ (\text{Before\_only\_c}(X) \wedge \text{After\_only\_d}(X) \wedge \text{Alternate}(X) \wedge \text{First\_a}(X) \wedge \text{Last\_b}(X))$$

- "X is a block of consecutive positions."

- "Before X there are only c's."

- "After X there are only d's."

- "a's and b's alternate in X."

- the first letter in X is an a and the last is a b.

Every regular language is expressible in MSO logic

Goal: given an arbitrary regular language L, construct an MSO sentence having L as language

We use: if L is regular, then there is a DFA A recognizing L.

Idea: construct a formula expressing
"the run of A on this word is accepting"

Every regular language is expressible in MSO logic

Proposition 7.12 *If $L \subseteq \Sigma^*$ is regular, then L is expressible in $MSO(\Sigma)$.*

Proof: Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA with $Q = q_0, \ldots, q_n$ and $\mathcal{L}(A) = L$. We construct a formula φ_A such that $w \models \varphi_A$ iff $w \in \mathcal{L}(A)$.

We start with some notations. Let $w = a_0 \ldots a_{m-1}$ be a word over Σ, and let

$$P_q = \{i \in \{0, \ldots, m-1\} \mid \delta(q_0, a_0 \ldots a_i) = q\} \ .$$

In words, $i \in P_q$ if A is in state q immediately *after* reading a_i. Then A accepts w iff $m - 1 \in \bigcup_{q \in F} P_q$.

Assume there is a formula $\text{Visits}(X_0, \ldots X_n)$ with free variables $X_0, \ldots X_n$ (we construct it later) such that $\mathfrak{I}(X_i) = P_{q_i}$ holds for *every* model $(w, \mathfrak{I})$ and for every $0 \leq i \leq n$. In words, $\text{Visits}(X_0, \ldots X_n)$ is only true when X_i takes the value P_{q_i} for every $0 \leq i \leq n$. Then $(w, \mathfrak{I})$ is a model of

$$\psi_A := \exists X_0 \ldots \exists X_n \ \text{Visits}(X_0, \ldots X_n) \wedge \exists x \left(\text{last}(x) \wedge \bigvee_{q_i \in F} x \in X_i) \right)$$

iff w has a last letter, and $w \in L$. So we can take

$$\varphi_A := \begin{cases} \psi_A & \text{if } q_0 \notin F \\ \psi_A \vee \forall x \ x < x & \text{if } q_0 \in F \end{cases}$$

It remains to construct Visits$(X_0, \ldots X_n)$. The sets P_q are the unique sets satisfying the following properties:

(a) $0 \in P_{q_{i_0}}$ for $q_{i_0} = \delta(q_0, a_0)$

(b) every position i belongs to exactly one P_q, i.e., the P_q's build a partition of the set positions, and

(c) if $i \in P_q$ and $\delta(q, a_{i+1}) = q'$ then $i + 1 \in P_{q'}$, i.e., the P_q's "respect" the transition function δ.

We express these properties through formulas. Formula for (a):

$$\text{Init}(X_{i_0}) = \exists x(\text{zero}(x) \wedge x \in X_{i_0})$$

Formula for (b):

$$\text{Partition}(X_0, \ldots, X_n) = \forall x \left(\bigvee_{i=0}^{n} x \in X_i \wedge \bigwedge_{\substack{i,j=0 \\ i \neq j}}^{n} (x \in X_i \rightarrow x \notin X_j) \right)$$

Formula for (c):

$$\text{Respect}(X_0, \ldots, X_n) = \forall x \forall y \left(y = x + 1 \rightarrow \bigvee_{\substack{a \in \Sigma \\ i,j \in \{0,\ldots,n\} \\ \delta(q_i, a) = q_j}} (x \in X_i \wedge Q_a(x) \wedge y \in X_j) \right)$$

Altogether we get

$$\text{Visits}(X_0, \ldots X_n) := \text{Init}(X_{i_0}) \wedge \text{Partition}(X_0, \ldots, X_n) \wedge \text{Respect}(X_0, \ldots, X_n)$$

Every language expressible in MSO logic is regular

An interpretation of a formula is a pair consisting of a word and assignments to the free first and second order variables (and perhaps to others).

$$\left(aab\,,\ \begin{array}{l} x \mapsto 0 \\ y \mapsto 2 \\ X \mapsto \{1,2\} \\ Y \mapsto \{0,1\} \end{array} \right) \qquad \left(ba\,,\ \begin{array}{l} x \mapsto 1 \\ y \mapsto 0 \\ X \mapsto \emptyset \\ Y \mapsto \{0\} \end{array} \right)$$

We encode interpretations of the free variables as words:

$$\left(aab\,,\ \begin{array}{l} x \mapsto 0 \\ y \mapsto 2 \\ X \mapsto \{1,2\} \\ Y \mapsto \{0,1\} \end{array} \right) \qquad \left(ba\,,\ \begin{array}{l} x \mapsto 1 \\ y \mapsto 0 \\ X \mapsto \emptyset \\ Y \mapsto \{0\} \end{array} \right)$$

	a	a	b
x	1	0	0
y	0	0	1
X	0	1	1
Y	1	1	0

	b	a
x	0	1
y	1	0
X	0	0
Y	1	0

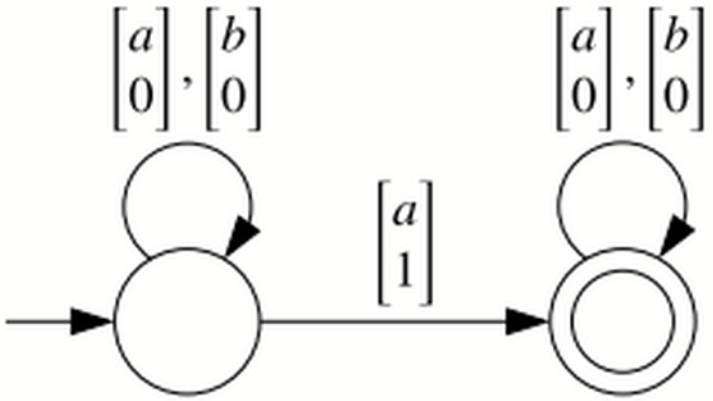
Definition 7.13 *Let φ be a formula with n free variables, and let $(w, \mathfrak{I})$ be an interpretation of φ. We denote by $\mathrm{enc}(w, \mathfrak{I})$ the word over the alphabet $\Sigma \times \{0,1\}^n$ described above. The language of φ is $\mathcal{L}(\varphi) = \{\mathrm{enc}(w, \mathfrak{I}) \mid (w, \mathfrak{I}) \models \varphi\}$.*

we prove by induction on the structure of φ that $\mathcal{L}(\varphi)$ is regular.

- $\varphi = Q_a(x)$. Then $\mathit{free}(\varphi) = x$, and the interpretations of φ are encoded as words over $\Sigma \times \{0, 1\}$. The language $\mathcal{L}(\varphi)$ is given by

$$\mathcal{L}(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ b_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i \in \{0, 1\} \text{ for every } i \in \{1, \ldots, k\}, \text{ and} \\ b_i = 1 \text{ for exactly one index } i \in \{1, \ldots, k\} \end{array} \right\}$$

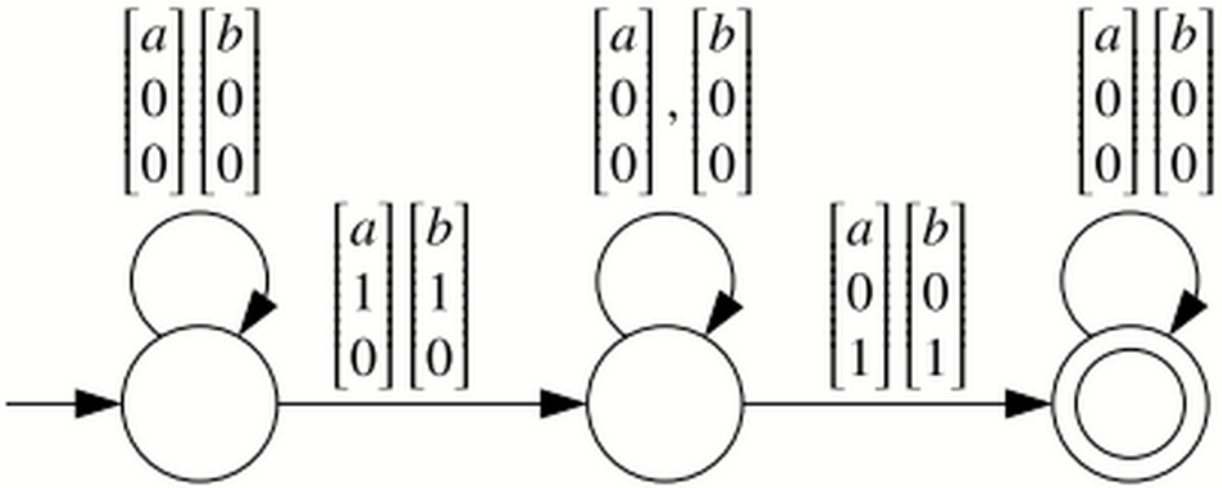
and is recognized by



- $\varphi = x < y$. Then $\mathit{free}(\varphi) = \{x, y\}$, and interpretations are encoded as words over $\Sigma \times \{0,1\}^2$. The language $\mathcal{L}(\varphi)$ is given by

$$\mathcal{L}(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i, c_i \in \{0,1\} \text{ for every } i \in \{1,\ldots,k\}, \\ b_i = 1 \text{ for exactly one index } i \in \{1,\ldots,k\}, \\ c_j = 1 \text{ for exactly one index } j \in \{1,\ldots,k\}, \text{ and} \\ i < j \end{array} \right\}$$

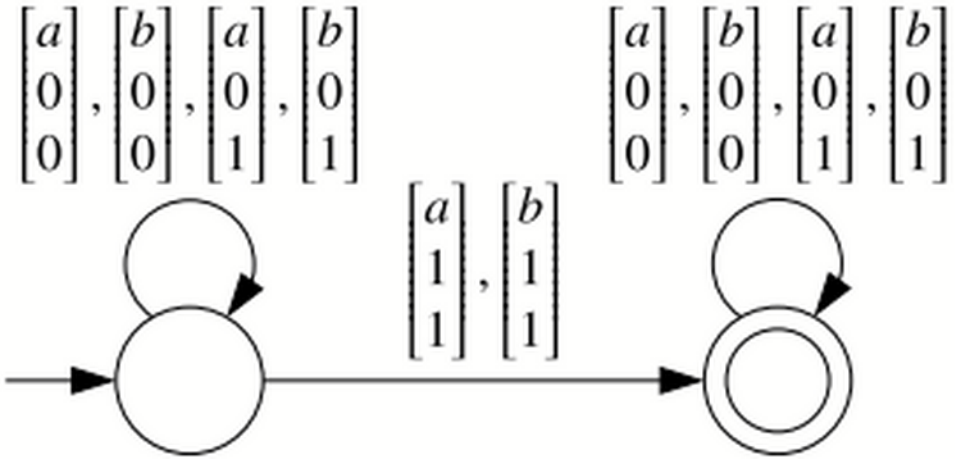
and is recognized by



- $\varphi = x \in X$. Then $free(\varphi) = \{x, X\}$, and interpretations are encoded as words over $\Sigma \times \{0,1\}^2$. The language $\mathcal{L}(\varphi)$ is given by

$$\mathcal{L}(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i, c_i \in \{0,1\} \text{ for every } i \in \{1,\ldots,k\}, \\ b_i = 1 \text{ for exactly one index } i \in \{1,\ldots,k\}, \text{ and} \\ \text{for every } i \in \{1,\ldots,k\}, \text{ if } b_i = 1 \text{ then } c_i = 1 \end{array} \right\}$$

and is recognized by



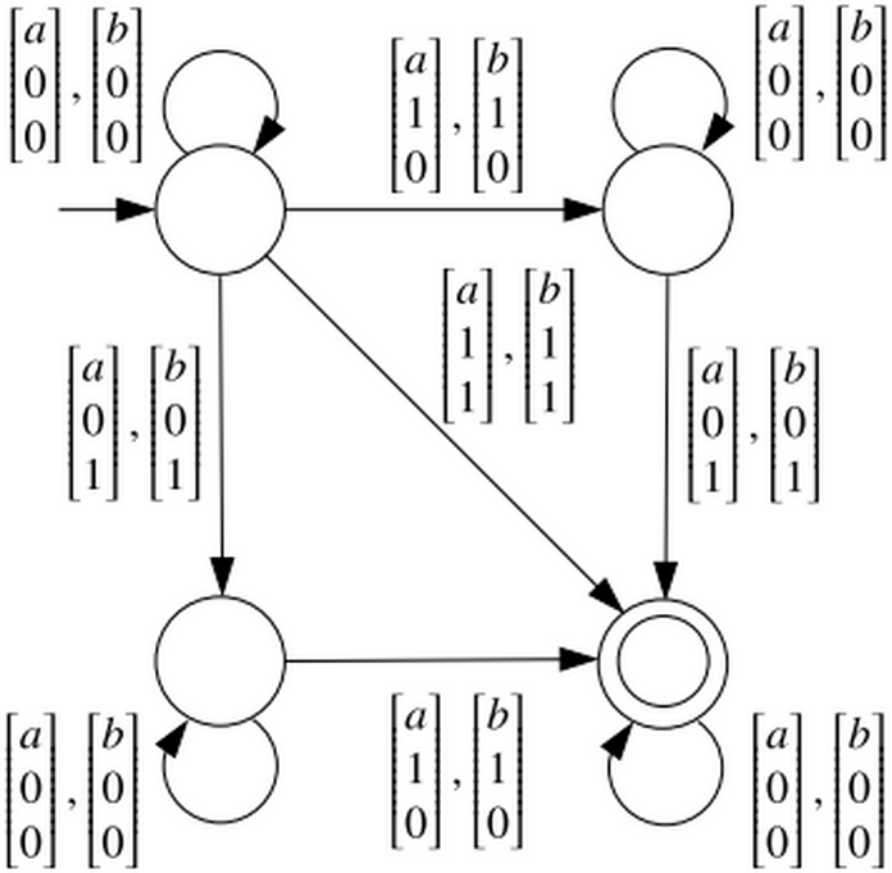
- $\varphi = \neg\psi$. Then $free(\varphi) = free(\psi)$, and by induction hypothesis there exists an automaton A_ψ s.t. $\mathcal{L}(A_\psi) = \mathcal{L}(\psi)$.

  Observe that $\mathcal{L}(\varphi)$ is *not* in general equal to $\overline{\mathcal{L}(\psi)}$. To see why, consider for example the case $\psi = Q_a(x)$ and $\varphi = \neg Q_a(x)$. The word

$$\begin{bmatrix} a \\ 1 \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix}$$

  belongs neither to $\mathcal{L}(\psi)$ nor $\mathcal{L}(\varphi)$, because it is not the encoding of any interpretation: the bitstring for x contains more than one 1. What holds is $\mathcal{L}(\varphi) = \overline{\mathcal{L}(\psi)} \cap Enc(\psi)$, where $Enc(\psi)$ is the language of the encodings of all the interpretations of ψ (whether they are models of ψ or not). We construct an automaton A_ψ^{enc} recognizing $Enc(\psi)$, and so we can take $A_\varphi = A_\psi \times_\cap A_\psi^{enc}$.

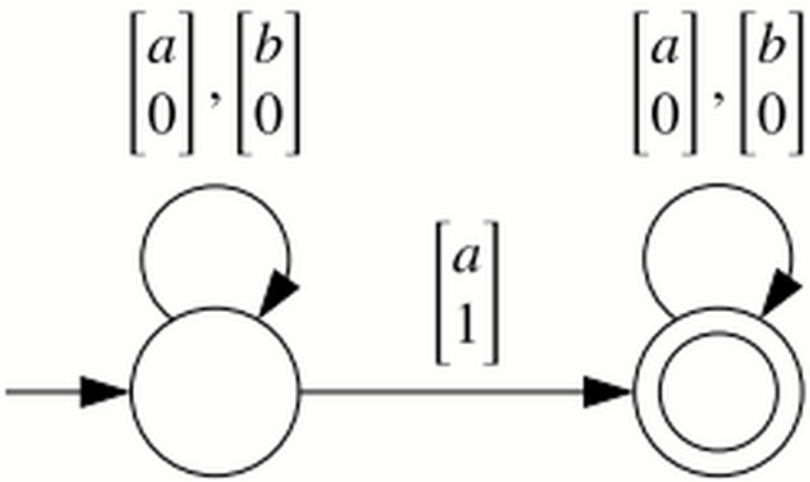
Assume ψ has k first-order variables. Then a word belongs to $Enc(\psi)$ iff each of its projections onto the 2nd, 3rd, ..., $(k+1)$-th component is a bitstring containing exactly one 1. As states of A_{ψ}^{enc} we take all the strings $\{0,1\}^k$. The intended meaning of a state, say state 101 for the case $k = 3$, is "the automaton has already read the 1's in the bitstrings of the first and third variables, but not yet read the 1 in the second." The initial and final states are 0^k and 1^k, respectively. The tran-



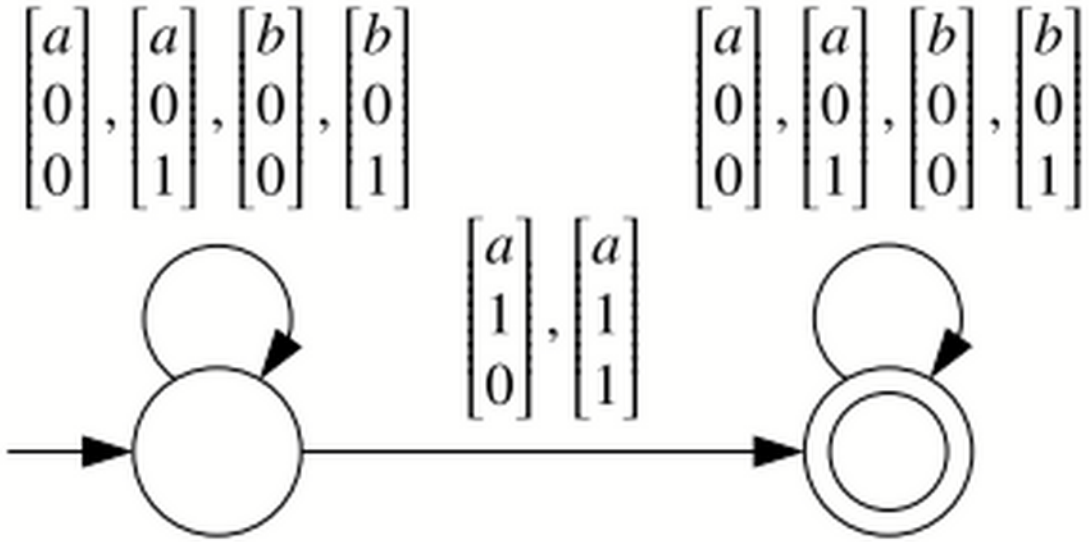
- $\varphi = \varphi_1 \wedge \varphi_2$. Then $free(\varphi) = free(\varphi_1) \cup free(\varphi_2)$, and by induction hypothesis there are automata A_{φ_i}, A_{φ_2} such that $\mathcal{L}(A_{\varphi_1}) = \mathcal{L}(\varphi_1)$ and $\mathcal{L}(A_{\varphi_2}) = \mathcal{L}(\varphi_2)$.

  If $free(\varphi_1) = free(\varphi_2)$, then we can take $A_\varphi = A_{\varphi_1} \times_\cap A_{\varphi_2}$. But this need not be the case. If $free(\varphi_1) \neq free(\varphi_2)$, then $\mathcal{L}(\varphi_1)$ and $\mathcal{L}(\varphi_2)$ are languages over different alphabets Σ_1, Σ_2, or over the same alphabet, but with different intended meaning, and we cannot just compute their intersection. For example, if $\varphi_1 = Q_a(x)$ and $\varphi_2 = Q_b(y)$, then both $\mathcal{L}(\varphi_1)$ and $\mathcal{L}(\varphi_2)$ are languages over $\Sigma \times \{0, 1\}$, but the second component indicates in the first case the value of x, in the second the value of y.

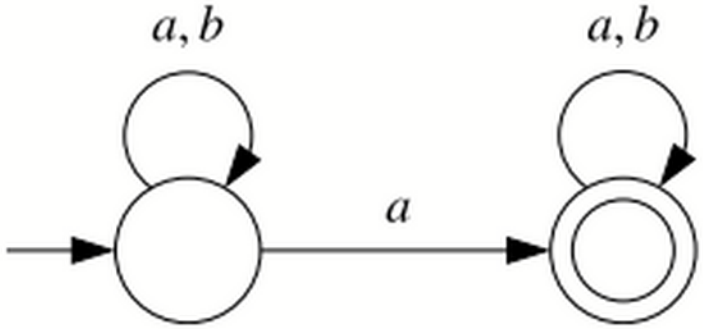
This problem is solved by extending $\mathcal{L}(\varphi_1)$ and $\mathcal{L}(A_{\varphi_2})$ to languages L_1 and L_2 over $\Sigma \times \{0, 1\}^2$. In our example, the language L_1 contains the encodings of all interpretations $(w, \{x \mapsto n_1, y \mapsto n_2\})$ such that the projection $(w, \{x \mapsto n_1\})$ belongs to $\mathcal{L}(Q_a(x))$, while L_2 contains the interpretations such that $(w, \{y \mapsto n_2\})$ belongs to $\mathcal{L}(Q_b(y))$. Now, given the automaton $A_{Q_a(x)}$ recognizing $\mathcal{L}(Q_a(x))$



we transform it into an automaton A_1 recognizing L_1



- $\varphi = \exists x\ \psi$. Then $free(\varphi) = free(\psi) \setminus \{x\}$, and by induction hypothesis there is an automaton A_ψ s.t. $\mathcal{L}(A_\psi) = \mathcal{L}(\psi)$. Define $A_{\exists\, x\psi}$ as the result of the projection operation, where we project onto all variables but x. The operation simply corresponds to removing in each letter of each transition of A_σ the component for variable x. For example, the automaton $A_{\exists x\ Q_a(x)}$ is obtained by removing the second components in the automaton for $A_{Q_a(x)}$ shown above, yielding



Observe that the automaton for $\exists x\ \psi$ can be nondeterministic even if the one for ψ is deterministic, since the projection operation may map different letters into the same one.

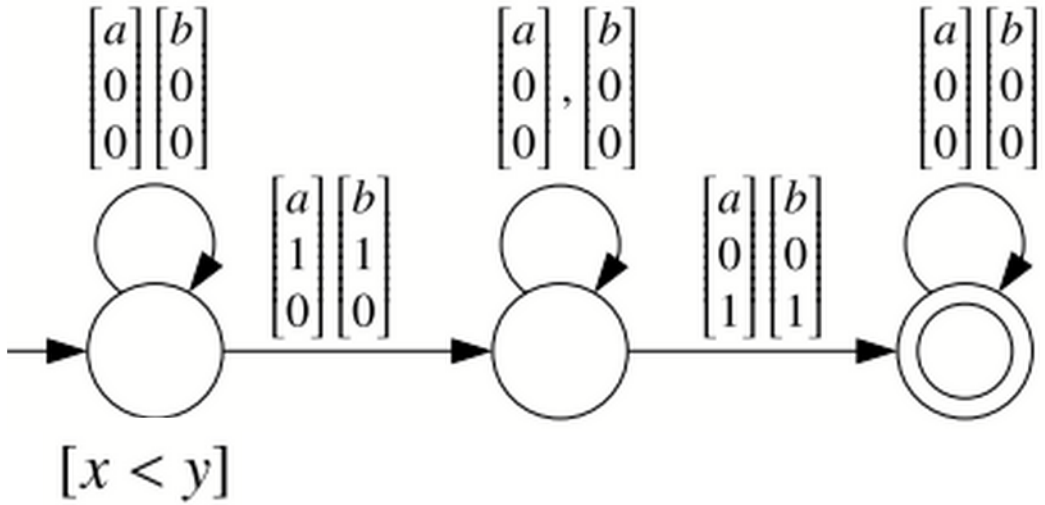
- $\varphi = \exists X\ \varphi$. We proceed as in the previous case.

The mega-example

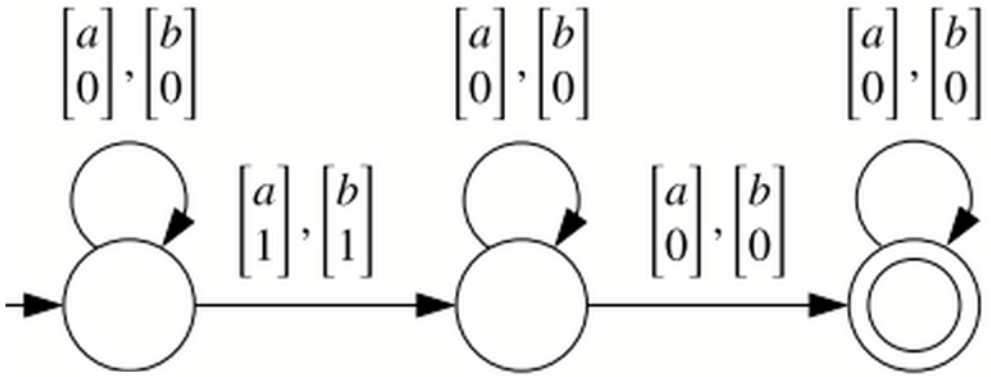
$$\varphi = \exists x\ (\text{last}(x) \wedge Q_b(x)) \ \wedge\ \forall x\ (\neg\text{last}(x) \rightarrow Q_a(x))$$

$$\psi = \exists x\ (\text{last}(x) \wedge Q_b(x)) \ \wedge\ \neg\exists x\ (\neg\text{last}(x) \wedge \neg Q_a(x))$$

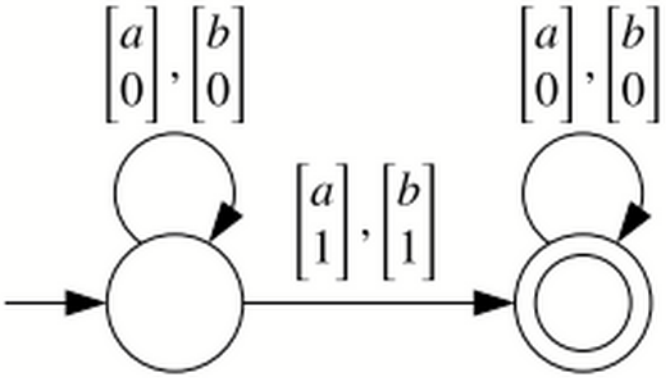
First, we compute an automaton for $\text{last}(x) \ = \ \neg\exists y\ x < y$.

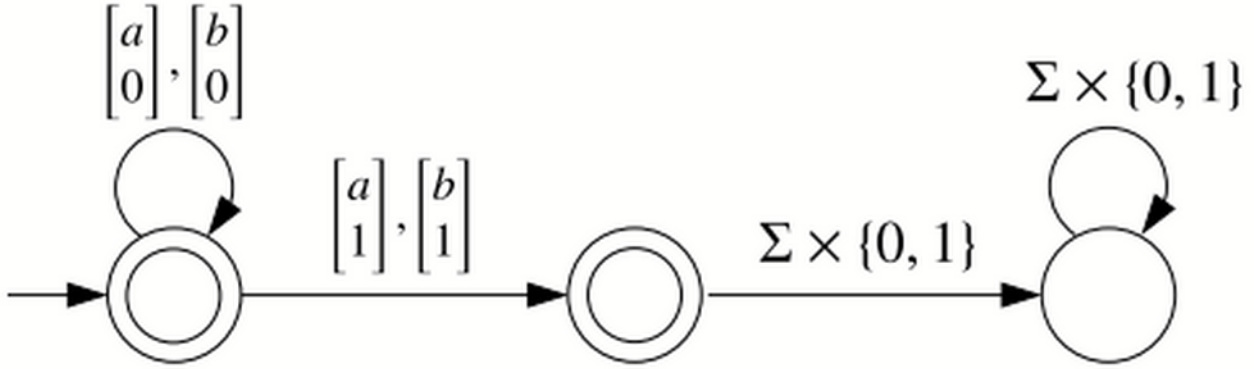


$[x < y]$

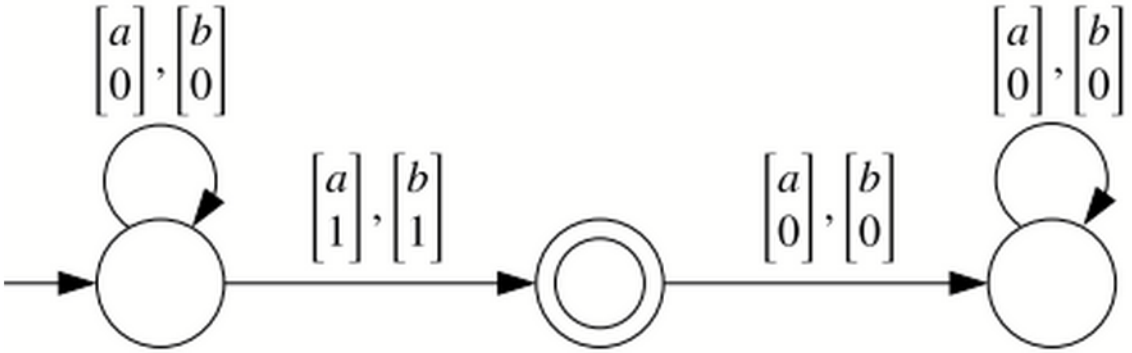


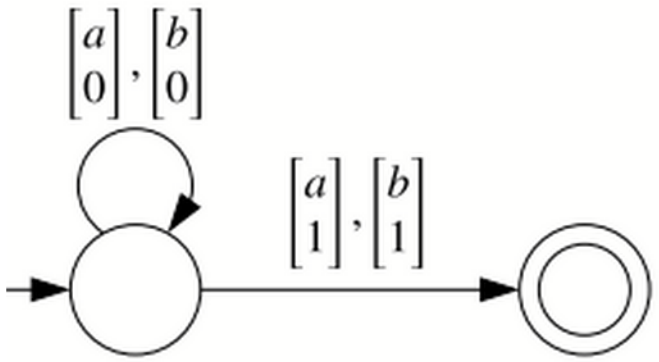
$[\exists y\ x < y]$





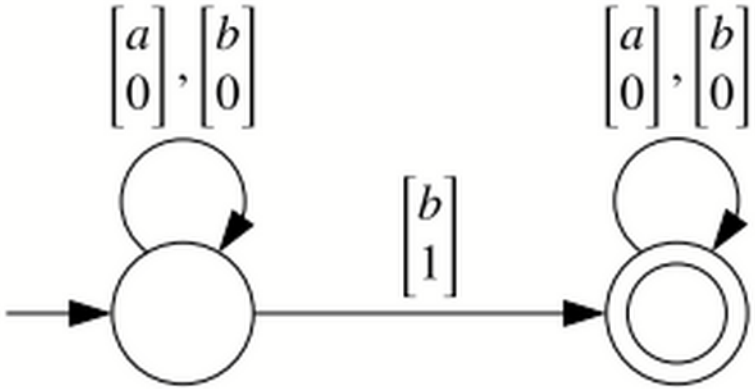
Enc($\exists y\ x < y$).



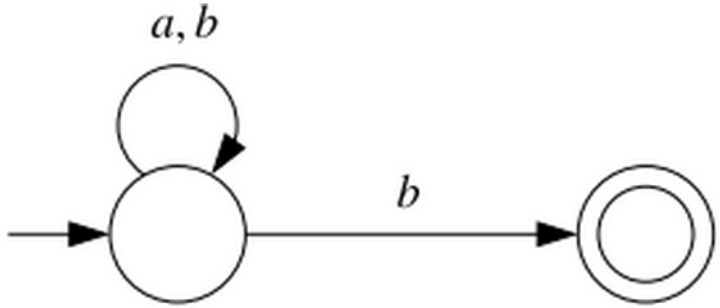


[last(x)]

Next we compute an automaton for $\exists x\ (\text{last}(x) \wedge Q_b(x))$

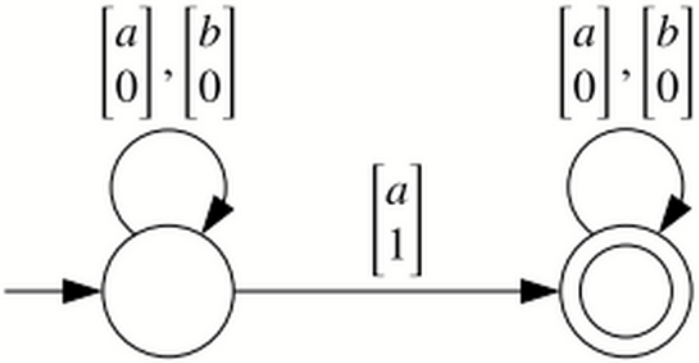


$[Q_b(x)]$

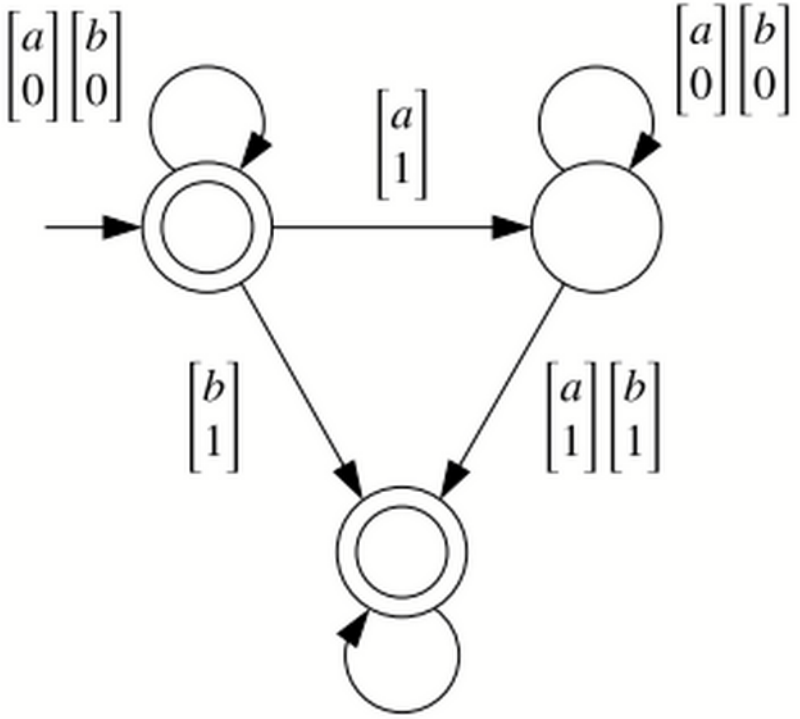


$[\exists x\ (\text{last}(x) \wedge Q_b(x))]$

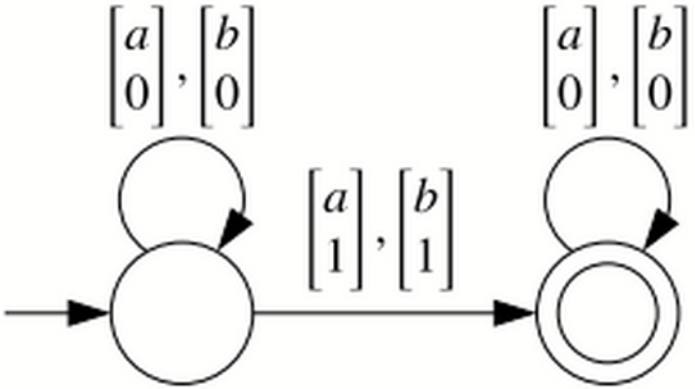
Now we compute an automaton for $\neg\exists x\ (\neg\text{last}(x) \wedge \neg Q_a(x))$.

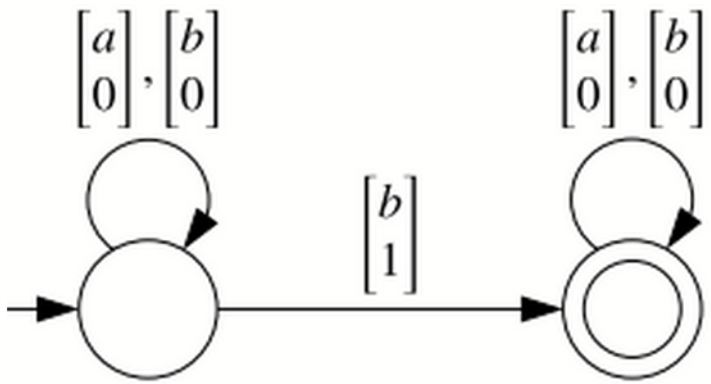


$[Q_a(x)]$

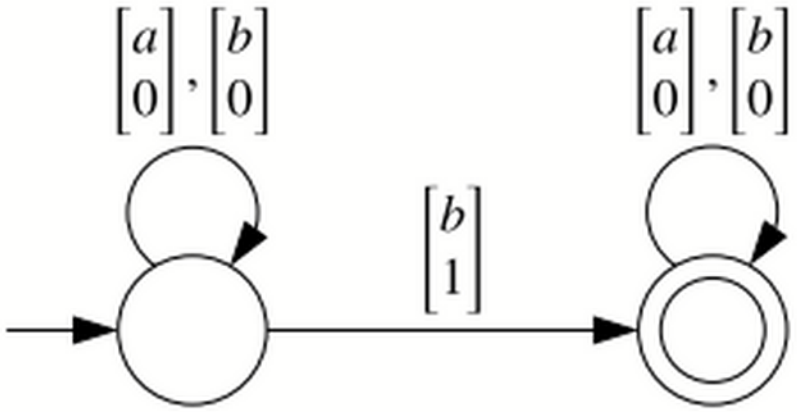


$\Sigma \times \{0, 1\}$

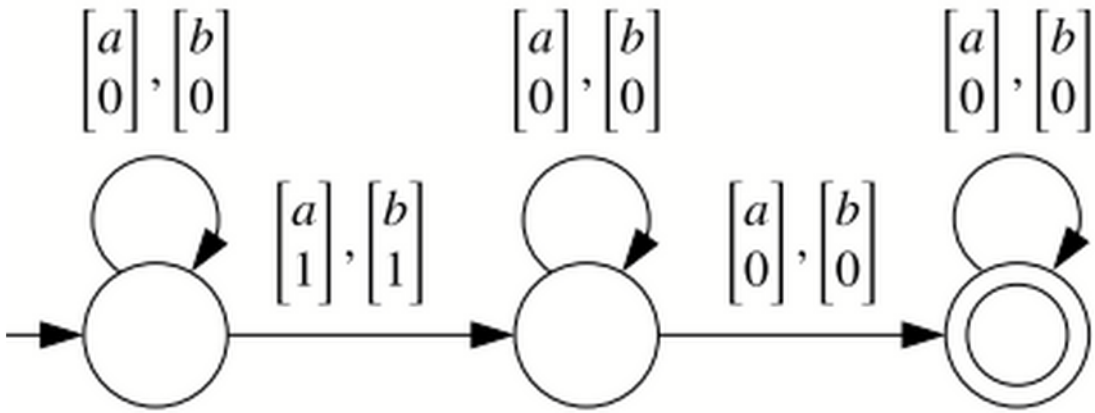




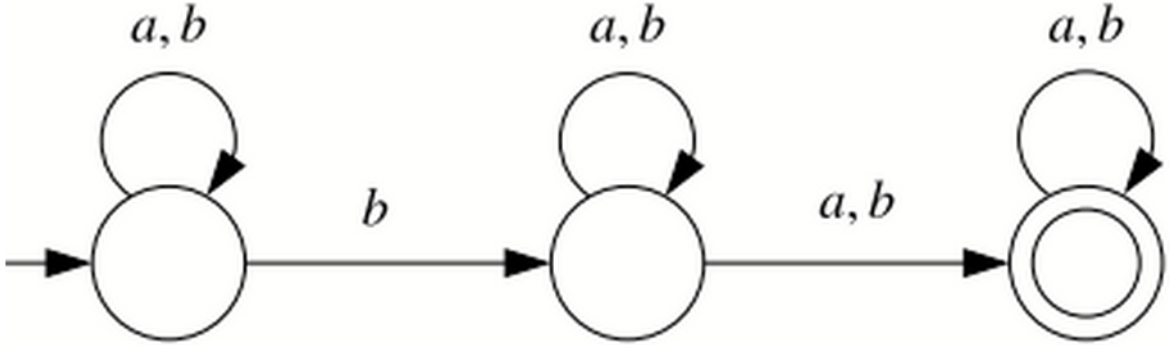
$[\neg Q_a(x)]$



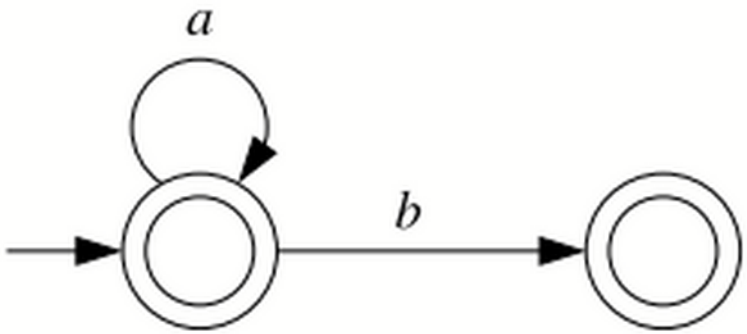
$[\neg Q_a(x)]$



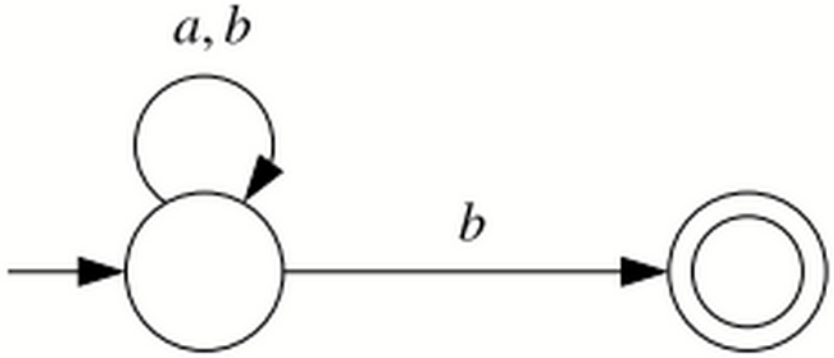
$[\neg last(x)]$



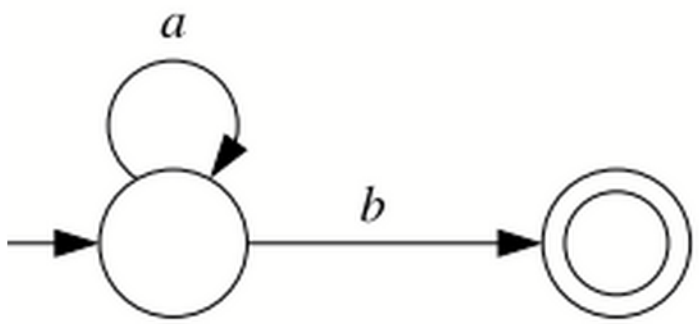
$[\exists x\ (\neg last(x) \wedge \neg Q_a(x))]$



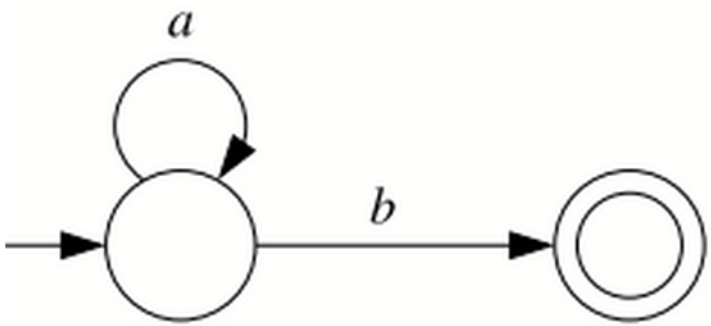
$[\neg \exists x\ (\neg last(x) \wedge \neg Q_a(x))]$



$[\exists x\ (\text{last}(x) \wedge Q_b(x))]$



$[\neg\exists x\ (\neg last(x) \wedge \neg Q_a(x))]$



$[\exists x\ (\text{last}(x) \wedge Q_b(x))\ \wedge\ \neg\exists x\ (\neg\text{last}(x) \wedge \neg Q_a(x))]$