Operations on sets: Implementation on DFAs

Member(x, X)	:	returns **true** if $x \in X$, **false** otherwise.
Complement(X)	:	returns $U \setminus X$.
Intersection(X, Y)	:	returns $X \cap Y$.
Union(X, Y)	:	returns $X \cup Y$.
Empty(X)	:	returns **true** if $X = \emptyset$, **false** otherwise.
Universal(X)	:	returns **true** if $X = U$, **false** otherwise.
Included(X, Y)	:	returns **true** if $X \subseteq Y$, **false** otherwise.
Equal(X, Y)	:	returns **true** if $X = Y$, **false** otherwise.

Assumption: each object encoded by one word, and viceversa.

Membership: trivial, linear in length of word.

Complement: trivial, swap final and non-final states.
Linear (or even constant) time.

Intersection(X, Y) : returns $X \cap Y$.
Union(X, Y) : returns $X \cup Y$.

SetDifference(X,Y) : returns X - Y, X\Y
Symmetric set difference: returns (X\Y) U (Y\X)

Op(X,Y,Z): returns (X U Y) \ Z

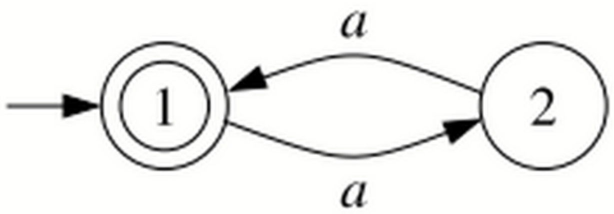
Pairing

Pairing

Definition 4.1 *Let* $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ *and* $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ *be DFAs. The* pairing $[A_1, A_2]$ *of* A_1 *and* A_2 *is the tuple* (Q, Σ, δ, q_0) *where:*

- $Q = \{ [q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2 \}$;
- $\delta = \{ ([q_1, q_2], a, [q_1', q_2']) \mid (q_1, a, q_1') \in \delta_1, (q_2, a, q_2') \in \delta_2 \}$;
- $q_0 = [q_{01}, q_{02}]$.

The run of $[A_1, A_2]$ *on a word of* Σ^* *is defined as for DFAs.*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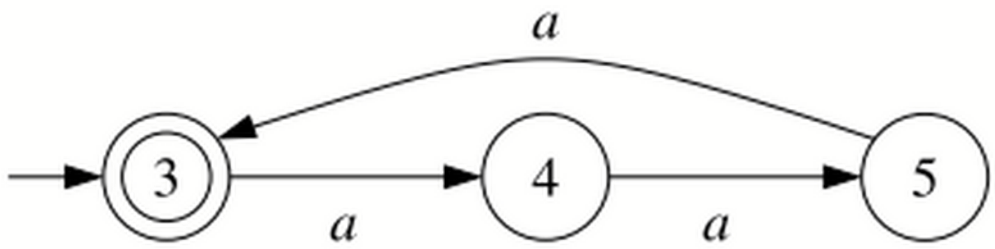
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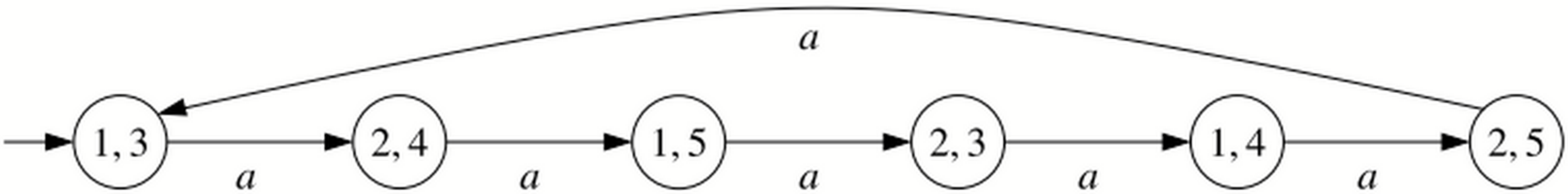
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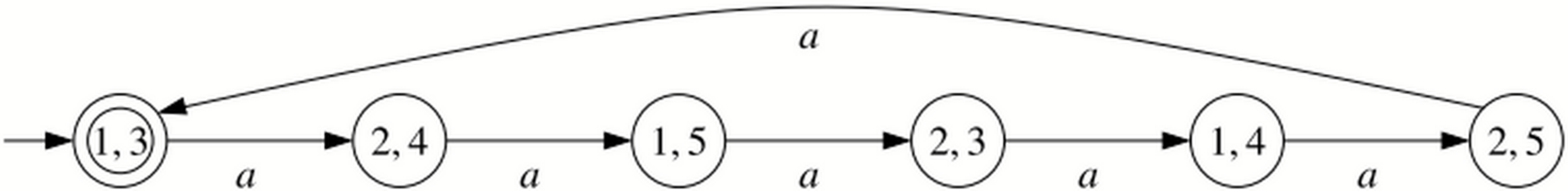
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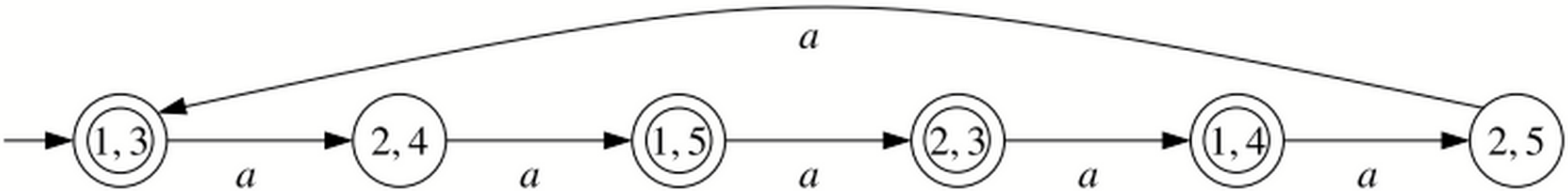
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Another example: even number of a's and even number of b's (and even number of c's ...)

Always remember:

an automaton for a regular language described as

``set of words satisfying .... some boolean combination of properties ...."

can be obtained by computing automata for the boolean properties, and then applying the composition operators.

A generic algorithm

$$L_1 \widehat{\odot} L_2 \quad = \quad \{w \in \Sigma^* \mid (w \in L_1) \odot (w \in L_2)\}$$

Language operation	$b_1 \odot b_2$
Union	$b_1 \vee b_2$
Intersection	$b_1 \wedge b_2$
Set difference ($L_1 \setminus L_2$)	$b_1 \wedge \neg b_2$
Symmetric difference ($L_1 \setminus L_2 \cup L_2 \setminus L_1$)	$b_1 \Leftrightarrow \neg b_2$

BinOp$[\odot](A_1, A_2)$

Input: DFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

Output: DFA $A = (Q, \Sigma, \delta, q_0, F)$ with $\mathcal{L}(A) = \mathcal{L}(A_1) \mathbin{\widehat{\odot}} \mathcal{L}(A_2))$

$Q \leftarrow \emptyset$; $F \leftarrow \emptyset$
$q_0 \leftarrow [q_{01}, q_{02}]$
$W \leftarrow \{q_0\}$
while $W \neq \emptyset$ **do**
    pick $[q_1, q_2]$ **from** W
    add $[q_1, q_2]$ **to** Q
    if $(q_1 \in F_1) \odot (q_2 \in F_2)$ **then add** $[q_1, q_2]$ **to** F
    for all $a \in \Sigma$ **do**
        $q'_1 \leftarrow \delta_1(q_1, a)$; $q'_2 \leftarrow \delta_2(q_2, a)$
        if $[q'_1, q'_2] \notin Q$ **then add** $[q'_1, q'_2]$ **to** W
        add $([q_1, q_2], a, [q'_1, q'_2])$ **to** δ
return $(Q, \Sigma, \delta, q_0, F)$

Complexity: the pairing of DFAs with n1 and n2 states has O(n1 n2) states.

Hence: for DFAs of size k1, k2, union, intersection, etc. can be caried out in time O(n1n2)

Language tests

Emptiness: no final states

Universality: only final states

Inclusion: L1 included in L2 iff L1 \ L2 is empty

Equality: L1 = L2 iff (L1\L2) U (L2\L1) is empty

InclDFA(A_1, A_2)

Input: DFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

Output: **true** if $\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2)$, **false** otherwise

$Q \leftarrow \emptyset$;

$W \leftarrow \{[q_{01}, q_{02}]\}$

while $W \neq \emptyset$ **do**

  pick $[q_1, q_2]$ **from** W

  add $[q_1, q_2]$ **to** Q

  if $(q_1 \in F_1)$ **and** $(q_2 \notin F_2)$ **then return false**

  for all $a \in \Sigma$ **do**

    $q'_1 \leftarrow \delta_1(q_1, a)$; $q'_2 \leftarrow \delta_2(q_2, a)$

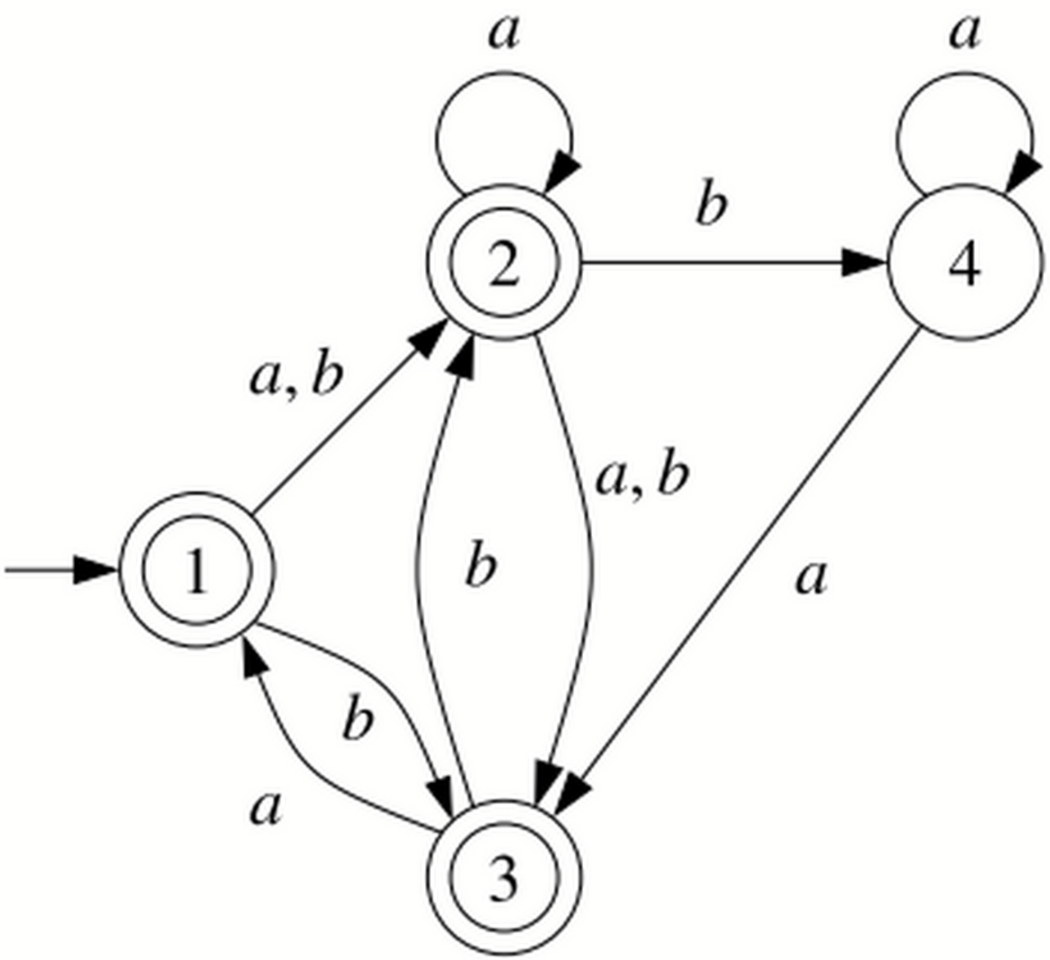
    if $[q'_1, q'_2] \notin Q$ **then add** $[q'_1, q'_2]$ **to** W

return true

Operations on sets: Implementation on NFAs

Member(x, X)	:	returns **true** if $x \in X$, **false** otherwise.
Complement(X)	:	returns $U \setminus X$.
Intersection(X, Y)	:	returns $X \cap Y$.
Union(X, Y)	:	returns $X \cup Y$.
Empty(X)	:	returns **true** if $X = \emptyset$, **false** otherwise.
Universal(X)	:	returns **true** if $X = U$, **false** otherwise.
Included(X, Y)	:	returns **true** if $X \subseteq Y$, **false** otherwise.
Equal(X, Y)	:	returns **true** if $X = Y$, **false** otherwise.

Membership



Prefix read	W
ϵ	$\{q_0\}$
a	$\{q_2\}$
aa	$\{q_2, q_3\}$
aaa	$\{q_1, q_2, q_3\}$
$aaab$	$\{q_2, q_3\}$
$aaabb$	$\{q_2, q_3, q_4\}$
$aaabba$	$\{q_1, q_2, q_3, q_4\}$

Mem[A](w)

Input: NFA $A = (Q, \Sigma, \delta, q_0, F)$, word $w \in \Sigma^*$,

Output: **true** if $w \in \mathcal{L}(A)$, **false** otherwise

$W \leftarrow \{q_0\}$;
while $w \neq \varepsilon$ **do**
  $U \leftarrow \emptyset$
  for all $q \in W$ **do**
    add $\delta(q, head(w))$ **to** U
  $W \leftarrow U$
  $w \leftarrow tail(w)$
return $(W \cap F \neq \emptyset)$

Complexity:
while loop executed |w| times
for loop executed at most |Q| times
each execution takes O(|Q|) time

Overall: O(|w||Q|^2) time

Complement

Swapping final and non-final states doesn't work:

Solution: determinize and then swap states.

Problem: Exponential blow-up in size!!
To be avoided whenever possible!!

No better way: there are NFAs with O(n) states such that the smallest NFA for the complement has O(2^n) states

(see the next exercise sheet)

Union and intersection

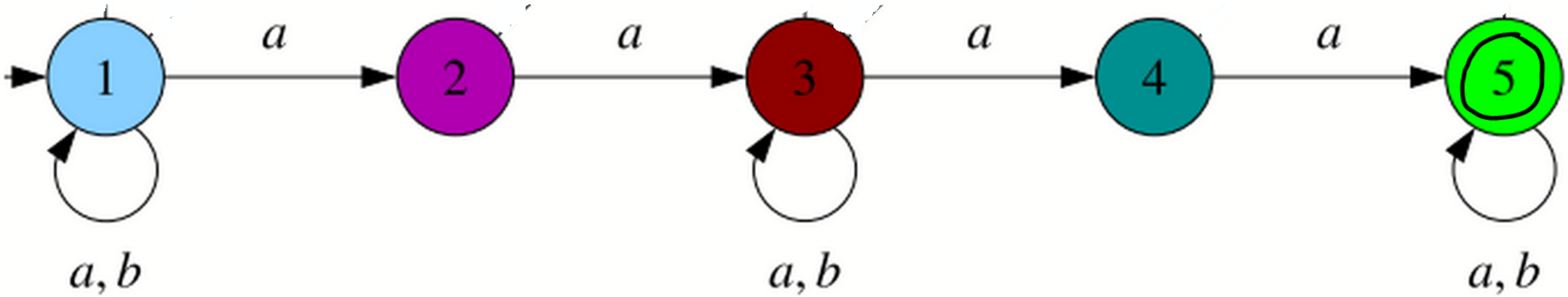
The pairing construction still works for union and intersection, with the same complexity, but not for set diference, or other non-monotonic operations.

There is a better construction for union, but not for intersection.

IntersNFA(A_1, A_2)
Input: NFA $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
Output: NFA $A_1 \cap A_2 = (Q, \Sigma, \delta, q_0, F)$ with $\mathcal{L}(A_1 \cap A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$

$Q \leftarrow \emptyset$; $F \leftarrow \emptyset$
$q_0 \leftarrow [q_{01}, q_{02}]$
$W \leftarrow \{ [q_{01}, q_{02}] \}$
while $W \neq \emptyset$ **do**
    pick $[q_1, q_2]$ **from** W
    add $[q_1, q_2]$ **to** Q
    if $q_1 \in F_1$ **and** $q_2 \in F_2)$ **then add** $[q_1, q_2]$ **to** F
    for all $a \in \Sigma$ **do**
        for all $q_1' \in \delta_1(q_1, a), q_2' \in \delta_2(q_2, a)$ **do**
        if $[q_1', q_2'] \notin Q$ **then add** $[q_1', q_2']$ **to** W
        add $([q_1, q_2], a, [q_1', q_2'])$ **to** δ
return $(Q, \Sigma, \delta, q_0, F)$

For the complexity, observe that in the worst case the algorithm must examine all pairs $[t_1, t_2]$ of transitions of $\delta_1 \times \delta_2$, but every pair is examined at most once. So the runtime is $\mathcal{O}(|\delta_1||\delta_2|)$.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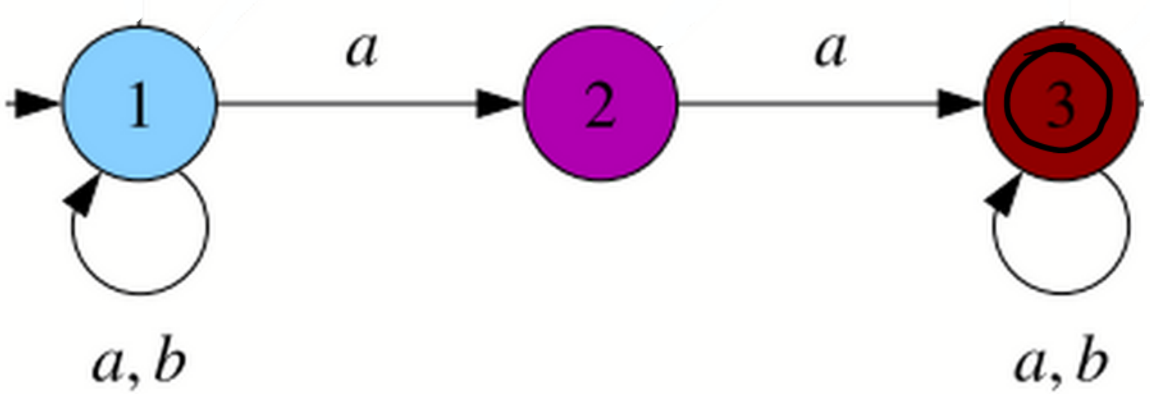
a, b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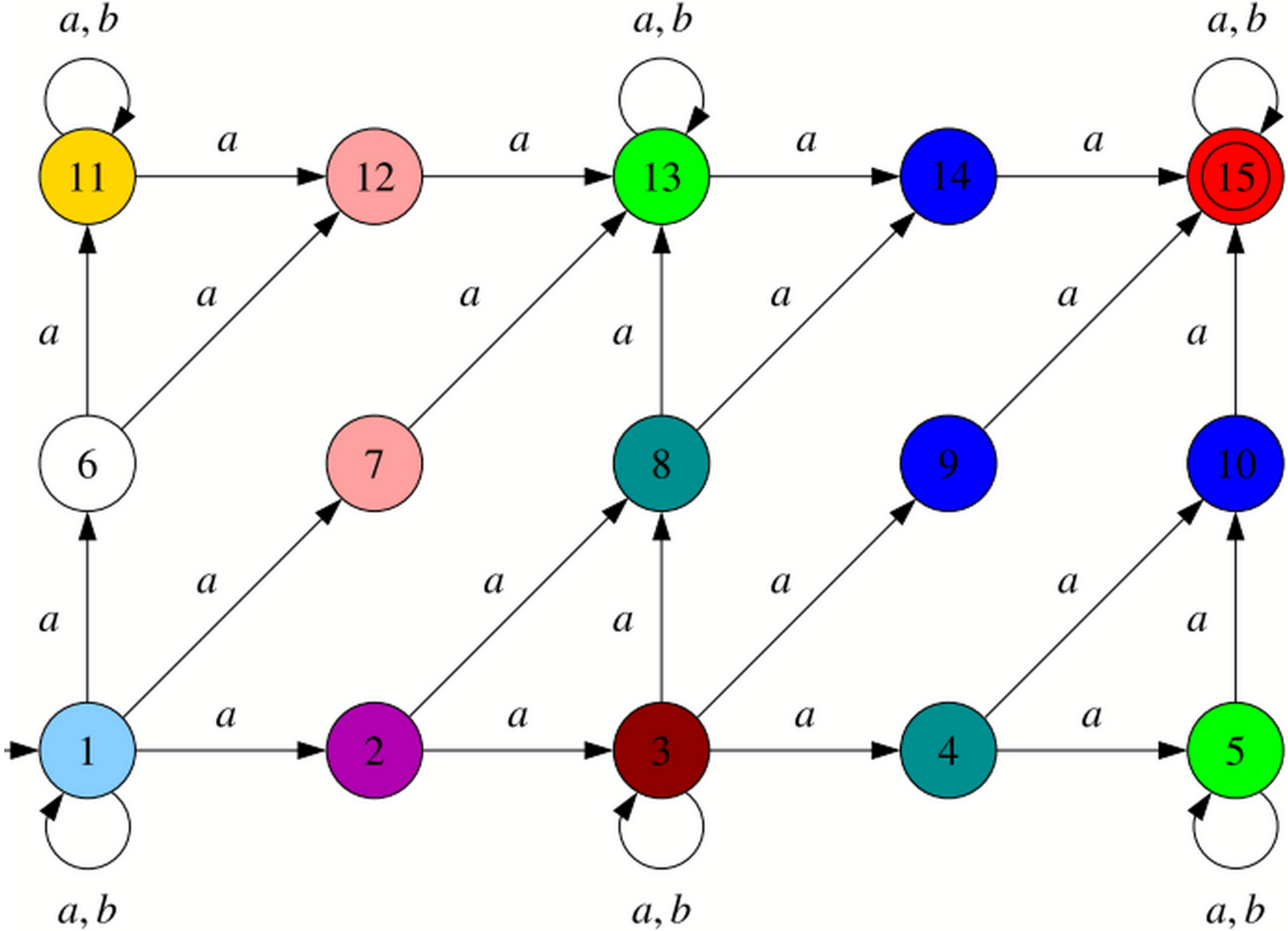

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15
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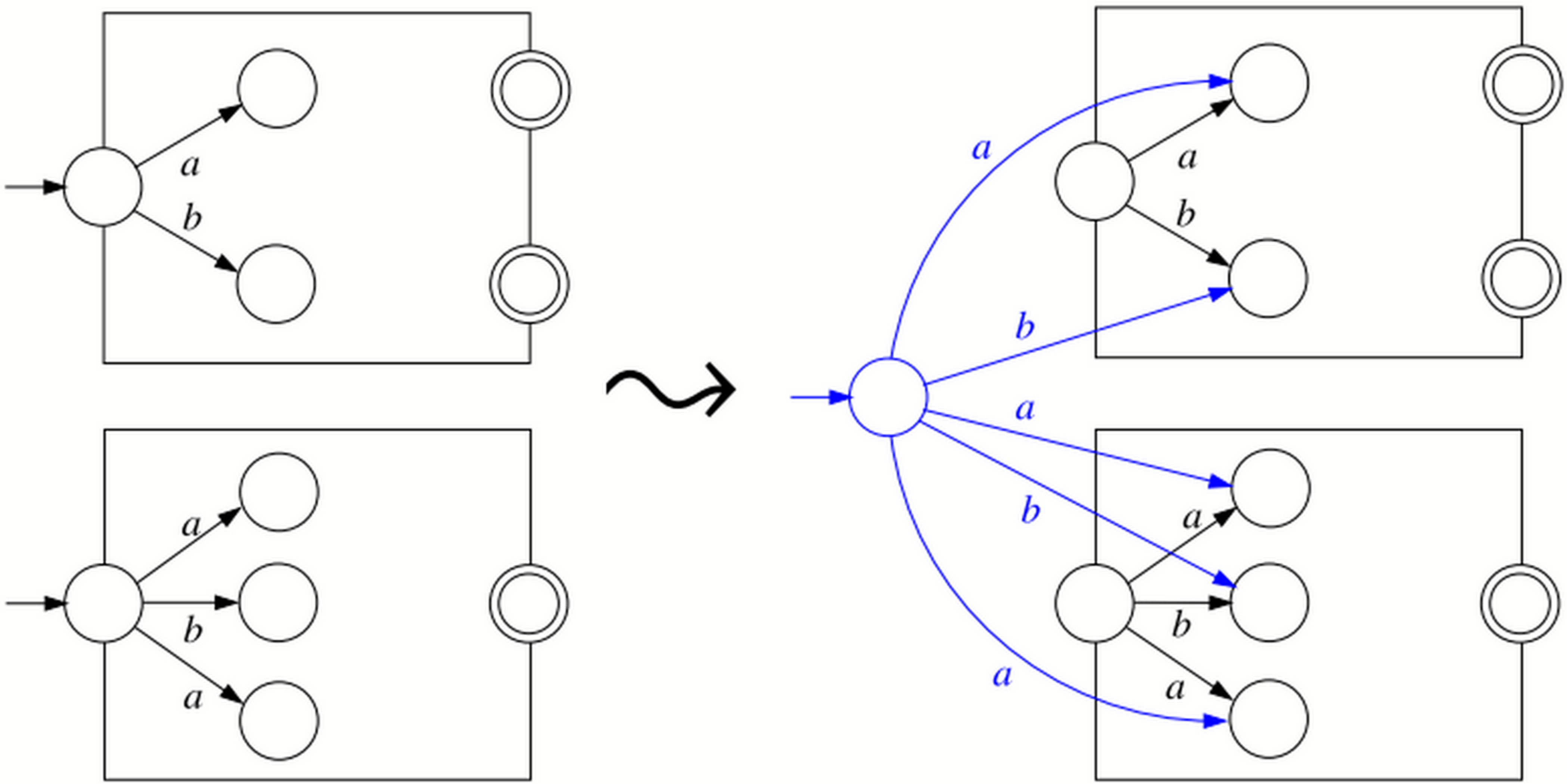
a, b

Union



UnionNFA(A_1, A_2)
Input: NFA $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
Output: NFA $A_1 \cup A_2$ with $\mathcal{L}(A_1 \cup A_2) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$

$Q \leftarrow Q_1 \cup Q_2 \cup \{q_0\}$; $F \leftarrow F_1 \cup F_2$
$\delta \leftarrow \delta_1 \cup \delta_2$
for all $(q_{01}, a, q) \in \delta_1$ **do**
    add (q_0, a, q_1) **to** δ
for all $(q_{02}, a, q) \in \delta_2$ **do**
    add (q_0, a, q_2) **to** δ
return $(Q, \Sigma, \delta, q_0, F)$

$\mathcal{O}(m_1 + m_2)$, where m_i is the number of transitions of A_i starting at q_{0i}.

Example showing that the pairing construction does not work for set difference:

- SetDiff(A,A) should always produce an NFA recognizing the empty language, but this is not the case.

Emptiness and universality

Exactly one of these two sentences is true:

NFA is empty iff every state is non-final

NFA is universal iff every state is final

Emptiness and universality

Exactly one of these two sentences is true:

NFA is empty iff every state is non-final

NFA is universal iff every state is final

Emptiness decidable in linear time.

Universality is PSPACE-complete.

Theorem 4.6 *The universality problem for NFAs is PSPACE-complete*

Proof: We only sketch the proof. To prove that the problem is in PSPACE, we show that it belongs to NPSPACE and apply Savitch's theorem. The polynomial-space nondeterministic algorithm for universality looks as follows. Given an NFA $A = (Q, \Sigma, \delta, q_0, F)$, the algorithm guesses a run of $B = \mathit{NFAtoDFA}(A)$ leading from $\{q_0\}$ to a non-final state, i.e., to a set of states of A containing no final state (if such a run exists). The algorithm only does not store the whole run, only the current state, and so it only needs linear space in the size of A.

We prove PSPACE-hardness by reduction from the acceptance problem for linearly bounded automata. A linearly bounded automaton is a deterministic Turing machine that always halts and only uses the part of the tape containing the input. A configuration of the Turing machine on an input of length k is coded as a word of length k. The run of the machine on an input can be encoded as a word $c_0\#c_1 \ldots \#c_n$, where the c_i's are the encodings of the configurations.

Let Σ be the alphabet used to encode the run of the machine. Given an input x, M accepts if there exists a word w of Σ^* satisfying the following properties:

(a) w has the form $c_0\#c_1 \ldots \#c_n$, where the c_i's are configurations;

(b) c_0 is the initial configuration;

(c) c_n is an accepting configuration; and

(d) for every $0 \leq i \leq n-1$: c_{i+1} is the successor configuration of c_i according to the transition relation of M.

The reduction shows how to construct in polynomial time, given a linearly bounded automaton M and an input x, an NFA $A(M, x)$ accepting all the words of Σ^* that do *not* satisfy at least one of the conditions (a)-(d) above. We then have

- If M accepts x, then there is a word $w(M, x)$ encoding the accepting run of M on x, and so $\mathcal{L}(A(M, x)) = \Sigma^* \setminus \{w(M, x)\}$.

- If M rejects x, then no word encodes an accepting run of M on x, and so $\mathcal{L}(A(M, x)) = \Sigma^*$.

So M accepts x if and only if $\mathcal{L}(A(M, x)) = \Sigma^*$, and we are done. □

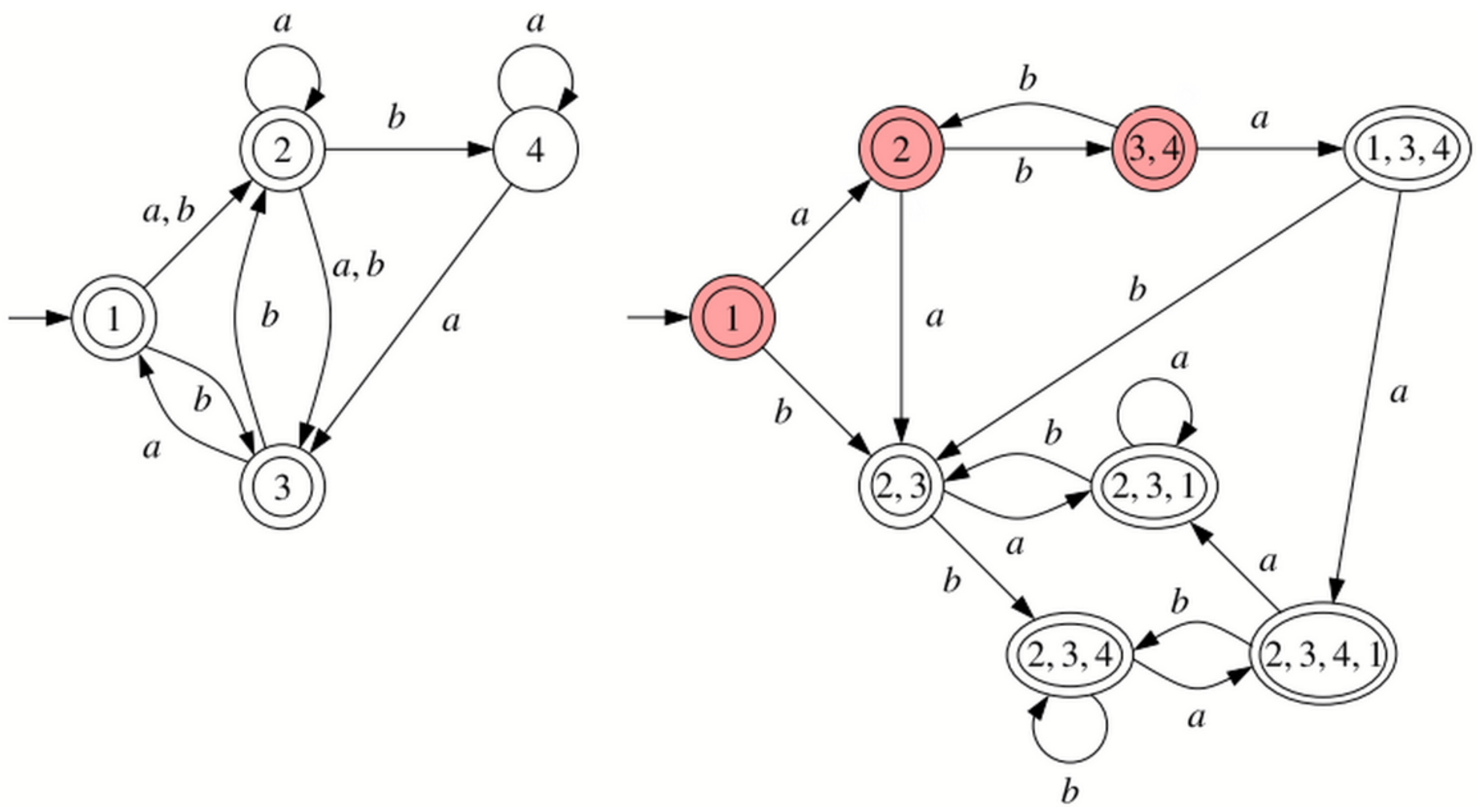
- Complement and check for emptiness
  Exponential complexity

- Improvements:
  (1) check for emptiness while complementing
      (on the fly check)
  (2) subsumption test

Definition 4.7 *Let A be a NFA, and let B = NFAtoDFA(A). A state Q' of B is* minimal *if no other state Q'' satisfies $Q'' \subset Q'$.*

Proposition 4.8 *Let A be a NFA, and let B = NFAtoDFA(A). A is universal iff every minimal state of B is final.*

Proof: Since A and B recognize the same language, A is universal iff B is universal. So A is universal iff every state of B is final. But a state of B is final iff it contains some final state of A, and so every state of B is final iff every minimal state of B is final. □

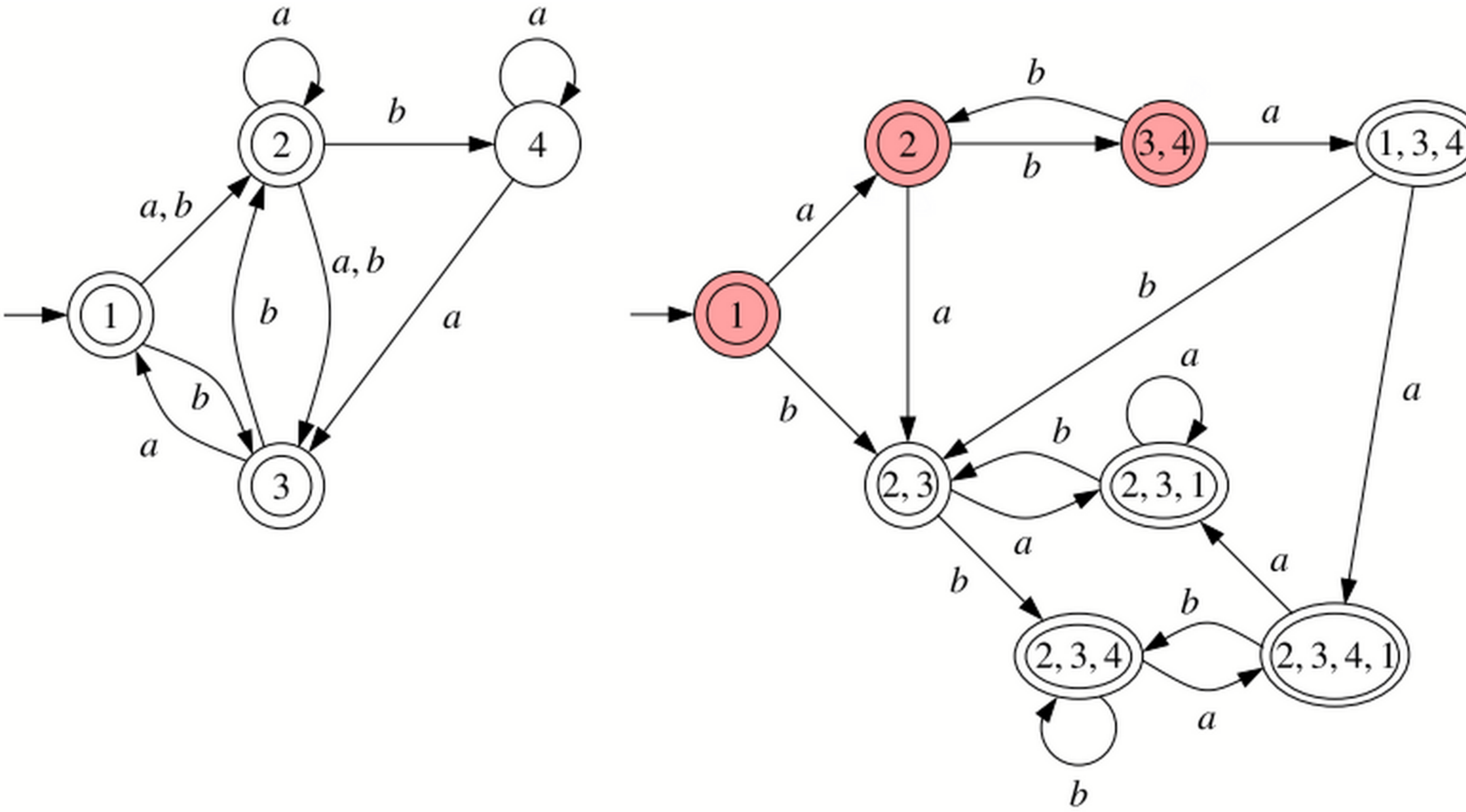


UnivNFA(A)

Input: NFA $A = (Q, \Sigma, \delta, q_0, F)$

Output: **true** if $\mathcal{L}(A) = \Sigma^*$, **false** otherwise

$\mathcal{Q} \leftarrow \emptyset$;
$\mathcal{W} \leftarrow \{ \{q_0\} \}$
while $\mathcal{W} \neq \emptyset$ **do**
  pick Q' **from** $\mathcal{W}$
  if $Q' \cap F = \emptyset$ **then return false**
  add Q' **to** $\mathcal{Q}$
  for all $a \in \Sigma$ **do**
    if $\mathcal{W} \cup \mathcal{Q}$ contains no $Q'' \subseteq \delta(Q', a)$ **then add** $\delta(Q', a)$ **to** $\mathcal{W}$
return true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2, 3
2, 3, 1
a
a
b
b
2, 3, 4
2, 3, 4, 1
a
b

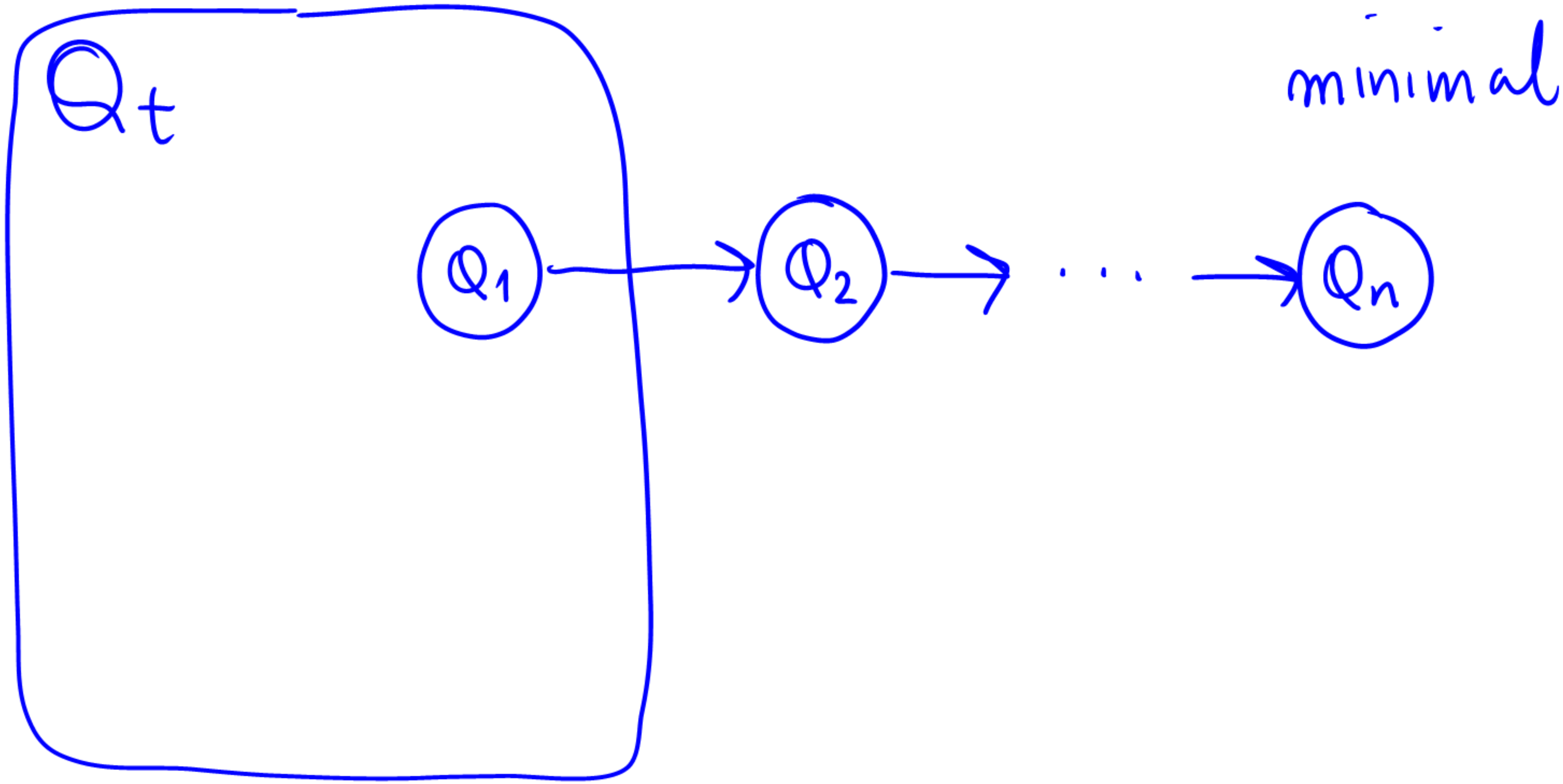
Is it correct?

By removing a non-minimal state we might be preventing the discovery of minimal states in the future!

Proposition 4.10 *Let $A = (Q, \Sigma, \delta, q_0, F)$ be a NFA, and let $B = NFAtoDFA(A)$. After termination of $UnivNFA(A)$, the set $\mathcal{Q}$ contains all minimal states of B.*

Proof: Let $\mathcal{Q}_t$ be the value of $\mathcal{Q}$ after termination of *UnivNFA*(A). We show that no path of B leads from a state of $\mathcal{Q}_t$ to a minimal state of B not in $\mathcal{Q}_t$. Since $\{q_0\} \in \mathcal{Q}_t$ and all states of B are reachable from $\{q_0\}$, it follows that every minimal state of B belongs to $\mathcal{Q}_t$.

Assume there is a path $\pi = Q_1 \xrightarrow{a_1} Q_2 \ldots Q_{n-1} \xrightarrow{a_n} Q_n$ of B such that $Q_1 \in \mathcal{Q}_t$, $Q_n \notin \mathcal{Q}_t$, and Q_n is minimal. Assume further that π is as short as possible. This implies $Q_2 \notin \mathcal{Q}_t$ (otherwise $Q_2 \ldots Q_{n-1} \xrightarrow{a_n} Q_n$ is a shorter path satisfying the same properties), and so Q_2 is never added to the worklist. On the other hand, since $Q_1 \in \mathcal{Q}_t$, the state Q_1 is eventually added to and picked from the worklist. When Q_1 is processed at line 7 the algorithm considers $Q_2 = \delta(Q_1, a_1)$, but does not add it to the worklist in line 8. So at that moment either the worklist or $\mathcal{Q}$ contain a state $Q_2' \subset Q_2$. This state is eventually added to $\mathcal{Q}$ (if it is not already there), and so $Q_2' \in \mathcal{Q}_t$. So B has a path $\pi' = Q_2' \xrightarrow{a_2} Q_3' \ldots Q_{n-1}' \xrightarrow{a_n} Q_n'$ for some states $Q_3', \ldots, Q_n'$. Since $Q_2' \subset Q_2$ we have $Q_2' \subset Q_2, Q_3' \subseteq Q_3, \ldots, Q_n' \subseteq Q_n$ (notice that we may have $Q_3' = Q_3$). By the minimality of Q_n, we get $Q_n' = Q_n$, and so π' leads from Q_2', which belongs to $\mathcal{Q}_t$, to Q_n, which is minimal and not in to $\mathcal{Q}_t$. This contradicts the assumption that π is a s short as possible. □



Q_t
Q_1
Q_2
Q_n
minimal

Inclusion and equality

Proposition 4.14 *The inclusion problem for NFAs is PSPACE-complete*

Proof: We first prove membership in PSPACE. Since PSPACE=co-PSPACE=NPSPACE, it suffices to give a polynomial space nondeterministic algorithm that decides non-inclusion. Given NFAs A_1 and A_2, the algorithm guesses a word $w \in \mathcal{L}(A_1) \setminus \mathcal{L}(A_2)$ letter by letter, maintaining the sets Q'_1, Q'_2 of states that A_1 and A_2 can reached by the word guessed so far. When the guessing ends, the algorithm checks that Q'_1 contains some final state of A_1, but Q'_2 does not.

Hardness follows from the fact that A is universal iff $\Sigma \subseteq \mathcal{L}(A)$, and so the universality problem, which is PSPACE-complete, is a subproblem of the inclusion problem.

□

$$L_1 \subseteq L_2 \text{ iff } L_1 \cap \overline{L_2} = \emptyset.$$

Concatenate four algorithms:

(1) determinize A2
(2) complement the result
(3) intersect it with A1, and
(4) check for emptiness

State of (3): pair (q, Q), where q in Q1 and Q subset of Q2

Easy optimizations:

- we only need the states of (3), not its transitions
- do not perform (1), then (2), then (3): construct directly the states of (3)
- check (4) while constructing (3)

Further optimization: subsumption test

Definition 4.12 *Let* A_1, A_2 *be NFAs, and let* $B_2 = NFAtoDFA(A_2)$. *A state* $[q_1, Q_2]$ *of* $[A_1, B_2]$ *is* minimal *if no other state* $[q_1', Q_2']$ *satisfies* $q_1' = q_1$ *and* $Q_2' \subset Q'$.

Proposition 4.13 *Let* $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1), A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ *be NFAs, and let* $B_2 = NFAtoDFA(A_2)$. $\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2)$ *iff every minimal state* $[q_1, Q_2]$ *of* $[A_1, B_2]$ *satisfying* $q_1 \in F_1$ *also satisfies* $Q_2 \cap F_2 \neq \emptyset$.

Proof: Since A_2 and B_2 recognize the same language, $\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2)$ iff $\mathcal{L}(A_1) \cap \overline{\mathcal{L}(A_2)} = \emptyset$ iff $\mathcal{L}(A_1) \cap \overline{\mathcal{L}(B_2)} = \emptyset$ iff $[A_1, B_2]$ has a state $[q_1, Q_2]$ such that $q_1 \in F_1$ and $Q_2 \cap F_2 = \emptyset$. But $[A_1, B_2]$ has some state satisfying this condition iff it has some minimal state satisfying it. $\square$

InclNFA(A_1, A_2)

Input: NFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

Output: **true** if $\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2)$, **false** otherwise

$Q \leftarrow \emptyset$;

$W \leftarrow \{\ [q_{01}, \{q_{02}\}]\ \}$

while $W \neq \emptyset$ **do**

    pick $[q_1, Q_2]$ **from** W

    if $(q_1 \in F_1)$ **and** $(Q_2 \cap F_2 = \emptyset)$ **then return false**

    add $[q_1, Q_2]$ **to** Q

    for all $a \in \Sigma, q_1' \in \delta_1(q_1, a)$ **do**

        $Q_2' \leftarrow \delta_2(Q_2, a)$

        if $\mathcal{W} \cup Q$ contains no $[q_1'', Q_2'']$ s.t. $q_1'' = q_1'$ and $Q_2'' \subseteq Q_2'$ **then**

            add $[q_1', Q_2']$ **to** $\mathcal{W}$

return true

Complexity:

- Let A1, A2, with n1,n2 states over a k-letter alphabet

- Without the subsumption test:
    - The while loop is executed at most n1 2^n2 times
    - The for-loop is executed at most O(n1) times
    - An execution of the loop takes O(n2^2) time
    - Overall: O(k n1^2 n2^2 2^n2) time

- With the subsumption test the worst-case complexity is higher. Exercise: give an upper bound.

Important special case:

- A1 is an NFA, A2 is a DFA
    - Complementing A2 is now easy
    - We get O(n1^2 n2) time

Equality: check inclusion in both directions.