Automata and Formal Languages – Homework 9

Due 19.12.2011.

Exercise 9.1

Let $k_1, k_2 \in \mathbb{N}_0$ be constants. Find a Presburger arithmetic formula, $\varphi(x, y)$, with free variables x and y such that $\mathcal{I} \models \varphi(x, y)$ iff $\mathcal{I}(x) \geq \mathcal{I}(y)$ and $\mathcal{I}(x) - \mathcal{I}(y) \equiv k_1 \pmod{k_2}$. Find a corresponding automaton for the case $k_1 = 0$ and $k_2 = 2$.

Exercise 9.2

Using the algorithms discussed in the lecture, construct a finite automaton for the Presburger formula

$$\exists y \,:\, x = 3y.$$

Exercise 9.3

You have seen in the lecture how to construct a finite automaton which represents all solutions for a given linear inequation

$$a_1x_1 + a_2x_2 + \ldots + a_kx_k \leq b \text{ with } a_1, a_2, \ldots, a_k, b \in \mathbb{Z} \qquad (*)$$

w.r.t. the least-significant-bit-first representation of $\mathbb{N}^k$ (see the algorithm PAtoDFA).

We may also use the most-significant-bit-first (msbf) representation of $\mathbb{N}^k$, e.g.,

$$\text{msbf}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \mathcal{L}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}^* \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

(a) Construct a finite automaton for the inequation $2x - y \leq 2$ w.r.t. the msbf representation.

(b) Try now to adapt the algorithm PAtoDFA to the msbf encoding.

(c) Recall that integers can be encoded as binary strings using two's complement: a binary string $s = b_0b_1b_2\ldots b_n$ is interpreted, assuming msbf, as the integer

$$-b_0 \cdot 2^n + b_1 \cdot 2^{n-1} + b_2 \cdot 2^{n-2} + \ldots + b_n \cdot 2^0.$$

In particular, s and $(b_0)^*s$ represent the same integer. This extends in the standard way to tuples of integers, e.g., the pair $(-3, 5)$ has the following encodings:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Construct an automaton accepting all (encondings of) *integer* solutions of the inequation $2x - y \leq 2$.
- Extend your algorithm from (b) such that the constructed automaton accepts all two's complement encodings of all integer solutions of $(*)$.