Technische Universität München **Winter term 2011/12**
I7
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Automata and Formal Languages – Homework 6

Due 24.11.2010.

Exercise 6.1

Transducers can be also seen as devices transforming input into output. Thus, they can capture the behaviour of simple programs.

Let P be the following program. The domain of the variables is assumed to be $\{0, 1\}$, and the initial value is assumed to be 0. Let $[i, x, y]$ denote the state of P that corresponds to the ith instruction, the value x in x, and the value y in y.

```
1   x ←?
2   write x
3   do
4     do
5       read y
6     until x = y
7     if eof then
8       write y
9       end
10    do
11      x ← x − 1
        or
12      y ← y + 1
13    until x ≠ y
14  until false
15  end
```

The initial state of the program P is $[1, 0, 0]$. By executing the first instruction, the program can move from state $[1, 0, 0]$ and either enter the state $[2, 0, 0]$ or the state $[2, 1, 0]$. In both cases, no input symbol is read and no output symbol is written during the transition between the states. Hence, the transition relation δ for P contains the transition rules $([1, 0, 0], (\varepsilon, \varepsilon), [2, 0, 0])$ and $([1, 0, 0], (\varepsilon, \varepsilon), [2, 1, 0])$. Similarly, by executing its second instruction, the program P must move from state $[2, 1, 0]$ and enter state $[3, 1, 0]$ while reading nothing and writing 1. Hence, δ contains also the transition rule $([2, 1, 0], (\varepsilon, 1)[3, 1, 0])$.

(a) Draw the ε-transducer that characterizes the program P.

(b) Can an overflow error occur?

(c) What are the possible values of x and y upon termination, i.e. reaching **end**?

(d) Can a pair of input 101 and output 01 occur?

(e) Let us have a regular set I of inputs and a regular set O of outputs. We may consider O to be the dangerous outputs that we want to avoid and we want to prove that using only I is safe, i.e. none of the dangerous outputs can occur. Describe an algorithm deciding given I and O whether there are $i \in I$ and $o \in O$ such that (i, o) is accepted.

Exercise 6.2

With transducers defined to be finite automata whose transitions are labeled by pairs of symbols $(a, b) \in \Sigma \times \Sigma$ only pairs of words $(a_0a_1 \dots a_l, b_0b_1 \dots b_l)$ of same length can be accepted. Consider therefore finite automata whose transitions are labeled by elements of $(\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\})$ instead, and call this class ε-transducers. As in the case of transducers, we say that an ε-transducer $\mathcal{A}$ accepts a word pair (w, w') if there is a run

$$q_0 \xrightarrow{(a_0,b_0)} q_1 \xrightarrow{(a_1,b_1)} \dots \xrightarrow{(a_n,b_n)} q_n \text{ with } a_i, b_i \in \Sigma \cup \{\varepsilon\}$$

such that $w = a_0a_1 \dots a_n$ and $w' = b_0b_1 \dots b_n$. Note that $|w| \leq n$ and $|w'| \leq n$. As usual, we write $\mathcal{L}(\mathcal{A})$ for the language of word pairs accepted by the ε-transducer $\mathcal{A}$.

(a) Construct ε-transducers $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{L}(\mathcal{A}_1) = \{(a^nb^m, c^{2n}) \mid n, m \geq 0\}$, and $\mathcal{L}(\mathcal{A}_2) = \{(a^nb^m, c^{2m}) \mid n, m \geq 0\}$.

(b) Apply the construction for the intersection of two finite automata to $\mathcal{A}_1$ and $\mathcal{A}_2$. Which language does the resulting ε-transducer accept?

(c) Show that there is no ε-transducer which accepts the language $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$.

Exercise 6.3

Let $\Sigma = \{0, 1\}$ and for $a, b \in \Sigma$ we define $a \cdot b$ to be the usual multiplication (also an analog of Boolean *and*) and $a \oplus b$ to be 0 if $a = b = 0$, and 1 otherwise (an analog of Boolean *or*).

Consider the following function $f : \Sigma^6 \to \Sigma$ defined by

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 \cdot x_2) \oplus (x_3 \cdot x_4) \oplus (x_5 \cdot x_6)$$

(a) Construct the minimal DFA recognizing $L_1 = \{x_1x_2x_3x_4x_5x_6 \mid f(x_1, x_2, x_3, x_4, x_5, x_6) = 1\}$.

(b) Construct the minimal DFA recognizing $L_2 = \{x_1x_3x_5x_2x_4x_6 \mid f(x_1, x_2, x_3, x_4, x_5, x_6) = 1\}$.

Note the difference in the ordering! Give an example of $w \in L_1 \setminus L_2$.

Note the diffrence in the size of the automata. More generally, consider

$$f(x_1, \dots, x_{2n}) = \bigoplus_{1 \leq k \leq n} (x_{2k-1} \cdot x_{2k})$$

and languages according to orderings $x_1x_2 \dots x_{2n-1}x_{2n}$ and $x_1x_3 \dots x_{2n-1}x_2x_4 \dots x_{2n}$. Although both languages encode "equivalent" information, their minimal automata differ vastly in the size: for the former ordering the size is linear in n, wheareas for the latter it is exponential.

Exercise 6.4

(a) Given two minimal DFAs accepting bounded languages L_1 and L_2 with words of length k, construct a minimal DFA accepting $L_1 \cup L_2$.

(b) For any language $L \subseteq \{0, 1\}^k$ of binary numbers of length k, we define $L + 1$ to be the language $\{w + 1 \bmod 2^k \mid w \in L\}$. Construct a minimal DFA accepting $L + 1$ from a minimal DFA accepting L.

(c) Let $A = (Q, \{0, 1\}, \delta, q_0, F)$ be a minimal DFA such that $\mathcal{L}(A)$ is a bounded language of binary numbers. What language is accepted by the automaton $A' = (Q, \{0, 1\}, \delta', q_0, F)$, where $\delta'(q, b) = \delta(q, 1 - b)$?