Solution to 2.5

(a) Let $\mathcal{A} = (Q, \Sigma_1, \delta, q_0, F)$ be a DFA. In the lecture, you have seen finite automata whose transitions are labeled by regular expressions, and not only by letters. We make use of this extension here. We construct from $\mathcal{A}$ a finite automoton $\mathcal{A}' = (Q, \Sigma_2, \delta', q_0, F)$ whose transitions are labeled by words over Σ_2, more precisely by the words $h(\Sigma_1) := \{h(a) \mid a \in \Sigma_1\}$. Note that this set is finite as Σ_1 is finite.

We then set for all $a \in \Sigma_1$

$$\delta'(q, h(a)) := \delta(q, a).$$

Otherwise δ' is defined to be the empty set.

This basically means that we apply h to the edge labels of the graph underlying $\mathcal{A}$, i.e., if $q \xrightarrow{a} q'$ in $\mathcal{A}$, then $q \xrightarrow{h(a)} q'$ in $\mathcal{A}'$.

We now show that $\mathcal{L}(\mathcal{A}') = h(\mathcal{L}(\mathcal{A})$.

- Consider some word $w = a_1 a_2 \dots a_n \in \mathcal{L}(\mathcal{A})$. Hence, there is an accepting run of $\mathcal{A}$ on w, i.e.,

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \dots \xrightarrow{a_n} q_n \text{ with } q_n \in F.$$

By definition of δ' we therefore have $q_i \xrightarrow{h(a_i)} q_{i+1}$ in $\mathcal{A}'$ for all transitions along this run, implying that $w' = h(w)$ is accepted by $\mathcal{A}'$. Hence, $h(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A}')$

- Assume thus that $w' \in \mathcal{L}(\mathcal{A}')$. Then there is some accepting run of $\mathcal{A}'$

$$q_0 \xrightarrow{u_1} q_1 \xrightarrow{u_2} q_2 \dots \xrightarrow{u_l} q_n \text{ with } q_n \in F \text{ and } u_i \in h(\Sigma_1).$$

By definition of δ' we find for every transition $q_i \xrightarrow{u_i} q_{i+1}$ of $\mathcal{A}'$ some $a_i \in \Sigma_1$ with $h(a_i) = u_i$ such that $q_i \xrightarrow{a_i} q_{i+1}$ in $\mathcal{A}$. By construction,

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \dots \xrightarrow{a_l} q_n \text{ with } q_n \in F$$

is a run of $\mathcal{A}$, in particular, it is an accepting run. So, $a_1 a_2 \dots a_l \in \mathcal{L}(\mathcal{A})$ and $h(a_1 a_2 \dots a_l) = w'$. Therefore, $\mathcal{L}(\mathcal{A}') \subseteq h(\mathcal{L}(\mathcal{A}))$.

(b) Now we are given a finite automaton $\mathcal{A}' = (Q, \Sigma_2, \delta', q_0, F)$ over the alphabet Σ_2', w.l.o.g. $\mathcal{A}'$ is deterministic, and we need to construct a finite automaton $\mathcal{A}$ accepting $h^{-1}(\mathcal{L}(\mathcal{A}'))$.

As $\mathcal{A}'$ is assumed to be deterministic, δ' can be thought of as a map from $Q \times \Sigma_2$ to Q and we may extend this map to $Q \times \Sigma_2^*$ in the natural way:

$$\delta'(q, \varepsilon) := q \text{ and } \delta'(q, a_1 a_2 \dots a_n) := \delta'(\dots \delta'(\delta'(q, a_1), a_2) \dots, a_n).$$

The idea now is that a transition of $\mathcal{A}$ labeled by $a \in \Sigma_1$ summarizes the behavior of $\mathcal{A}'$ when reading the word $h(a)$.

Hence set

$$\delta(q, a) := \delta'(q, h(a)) \text{ for all } a \in \Sigma_1.$$

We claim that $\mathcal{A} = (Q, \Sigma_1, \delta, q_0, F)$ then accepts exactly $h^{-1}(\mathcal{L}(\mathcal{A}'))$.

- $\mathcal{L}(\mathcal{A}) \subseteq h^{-1}(\mathcal{L}(\mathcal{A}'))$:

  Choose some $w = a_1 a_2 \dots a_n \in \mathcal{L}(\mathcal{A})$, i.e.,

$$F \ni \delta(q_0, w) = \delta(\dots, \delta(\delta(q_0, a_1), a_2) \dots, a_n) \overset{\text{by Induction}}{=} \delta'(\dots, \delta'(\delta'(q_0, h(a_1)), h(a_2)) \dots, h(a_n)) = \delta'(q_0, h(w)).$$

  So, $h(w) \in \mathcal{L}(\mathcal{A}')$, i.e., $w \in h^{-1}(\mathcal{L}(\mathcal{A}'))$.

- $\mathcal{L}(\mathcal{A}) \supseteq h^{-1}(\mathcal{L}(\mathcal{A}'))$:

  Let $w = a_1 a_2 \dots a_n \in h^{-1}(\mathcal{L}(\mathcal{A}'))$, i.e., $h(w) \in \mathcal{L}(\mathcal{A}')$, i.e.,

$$F \ni \delta'(q_0, h(w)) = \delta'(\dots, \delta'(\delta'(q_0, h(a_1)), h(a_2)) \dots, h(a_n)) \overset{\text{by Induction}}{=} \delta(\dots, \delta(\delta(q_0, a_1), a_2) \dots, a_n) = \delta(q_0, w).$$

  So, $w \in \mathcal{L}(\mathcal{A})$.

(c) Set $L := \{(01^k 2)^n 3^n \mid k, n \geq 0\}$.

Let $h,\ \{0, 1, 2, 3\}^* \to \{0, 1\}^*$ be the homomorphism uniquely determined by

$$h(0) = 0, h(1) = \varepsilon, h(2) = \varepsilon, h(3) = 1.$$

Then $h(L) = \{0^n 1^n \mid n \geq 0\}$.

So, if L was regular, i.e., if there was some finite automaton $\mathcal{A}$ with $L = \mathcal{L}(\mathcal{A})$, then by the preceding results there would also be a finite automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}') = \{0^n 1^n \mid n \geq 0\}$. Contradiction.