Technische Universität München **Winter term 2009/10**
I7
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Automata and Formal Languages – Homework 1

Due 29.10.2009.

Exercise 1.1

In the lecture, it was shown how to obtain from a finite automaton a regular expression representing the same language by iteratively eliminating states of the automaton. In this exercise we will see that eliminating states corresponds to eliminating variables from a linear system of equations over the algebraic structure given by the set of languages 2^{Σ^*} with set union, language concatenation and the Kleene star as operations.

(a) You might remember Arden's Lemma from the introductionary course on formal language theory. Arden's Lemma says:

Given two languages $A, B \subseteq \Sigma^*$ with $\varepsilon \notin A$ then there is a unique language $X \subseteq \Sigma^*$ satisfying $X = AX \cup B$ and this language is given by A^*B.

Prove Arden's Lemma.

(b) Solve the following system given in two variables X, Y over $\Sigma = \{a, b, c, d, e, f\}$.

$$\begin{aligned} X &= \{a\}X \cup \{b\}Y \cup \{c\} \\ Y &= \{d\}X \cup \{e\}Y \cup \{f\}. \end{aligned}$$

Hint: Consider X as a constant language and solve the equation for Y using Arden's Lemma.

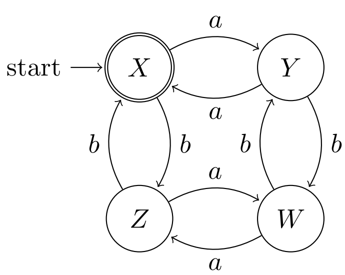
We can associate with any finite automaton $\mathcal{A} = (Q, \Sigma, \delta, q_I, F)$ a linear equation system as follows:

We take as variables the states of the automaton. For every state X we then have the equation

$$X = \left(\bigcup_{Y \in Q} \bigcup_{a \in \Sigma} \{a\}\delta(X, a, Y)Y \right) \cup \{\varepsilon\}\chi_F(X),$$

with $\delta(X, a, Y) = \{\varepsilon\}$ if $Y \in \delta(X, a)$, $\delta(X, a, Y) = \emptyset$ otherwise; and $\chi_F(X) = \{\varepsilon\}$ if $X \in F$, $\chi_F(X) = \emptyset$ otherwise.

(c) Consider the automaton depicted on the left and the equation system we obtain from it (shown on the right):



$$\begin{aligned} X &= \{a\}Y \cup \{b\}Z \cup \{\varepsilon\} \\ Y &= \{a\}X \cup \{b\}W \\ Z &= \{b\}X \cup \{a\}W \\ W &= \{b\}Y \cup \{a\}Z \end{aligned}$$

Calculate the solution of this linear system by iteratively eliminating variables. Start with Y, then eliminate Z, finally W.

Compare the solution you obtain to the regular expression obtained in the script for this automaton.

Exercise 1.2

Let Σ be an alphabet. We define the operator $|| : \Sigma^* \times \Sigma^* \to 2^{\Sigma^*}$ as follows:

$$v||u := u||v, \quad u||\varepsilon := \{u\}, \quad au||bv := \{aw \mid w \in u||bv\} \cup \{bw \mid w \in au||v\} \text{ for } a, b \in \Sigma, u, v \in \Sigma^*.$$

Examples:

$$b||d = \{bd, db\}, \quad ab||d = \{abd, adb, dab\}, \quad ab||cd = \{cabd, acbd, abcd, cadb, acdb, cdab\}.$$

- Show that for two regular languages $L_1, L_2 \subseteq \Sigma^*$ their interleaving

$$L_1||L_2 := \bigcup_{u \in L_1, v \in L_2} u||v$$

is also regular.

Exercise 1.3

Let Σ_1, Σ_2 be two alphabets. A map $h : \Sigma_1^* \to \Sigma_2^*$ is called a *homomorphism* if it respects the empty word and concatenation, i.e.,

$$h(\varepsilon) = \varepsilon \text{ and } h(w_1 w_2) = h(w_1)h(w_2) \text{ for all } w_1, w_2 \in \Sigma_1^*.$$

Assume that $h : \Sigma_1^* \to \Sigma_2^*$ is a homorphism. Note that h is completely determined by its values on Σ_1.

(a) Let $\mathcal{A}$ be a finite automaton over the alphabet Σ_1. Describe how to constuct a finite automaton accepting the language

$$h(\mathcal{L}(\mathcal{A})) := \{h(w) \mid w \in \mathcal{L}(\mathcal{A})\}.$$

(b) Let $\mathcal{A}'$ be a finite automaton over the alphabet Σ_2. Describe how to construct a finite automaton accepting the language

$$h^{-1}(\mathcal{L}(\mathcal{A}')) := \{w \in \Sigma_1^* \mid h(w) \in \mathcal{L}(\mathcal{A}')\}.$$

(c) Recall that the language $\{0^n 1^n \mid n \in \mathbb{N}\}$ is context free, but not regular. Use the preceding two results to show that $\{(01^k 2)^n 3^n \mid k, n \in \mathbb{N}\}$ is also not regular.

Exercise 1.4

For L_1, L_2 regular languages over an alphabet Σ, the *left quotient* of L_1 by L_2 is defined by

$$L_2 \diagdown L_1 := \{v \in \Sigma^* \mid \exists u \in L_2 : \; uv \in L_1\}$$

(a) Use the fact that regular languages are closed under homomorphisms, inverse homomorphisms, concatenation and intersection to prove they are closed under quotienting.

(b) Given finite automata $\mathcal{A}_1, \mathcal{A}_2$, construct an automaton $\mathcal{A}$ such that

$$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_2) \diagdown \mathcal{L}(\mathcal{A}_1)$$

(c) Is there any difference when taking the *right quotient* $L_1 \diagup L_2 := \{u \in \Sigma^* \mid \exists v \in L_2 : \; uv \in L_1\}$?

Exercise 1.5

Let L_1, L_2 be regular languages. Determine the inclusion relation between the following languages:

- L_1
- $(L_1 \diagup L_2).L_2$
- $(L_1.L_2) \diagup L_2$