Exercise 4.2

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be an NFA.

We introduce a "game" on $\mathcal{A}$: We have two players, called "refuter" (short: $\perp$) and "prover" (short: $\top$). A play $\pi = \{s_0, t_0\}\{s_1, t_1\}\ldots$ of the two players on $\mathcal{A}$ is a sequence of (unordered) pairs of states. The initial pair $\{s_0, t_0\}$ can be chosen arbitrarily, but the choice of a successor pair $\{s_{i+1}, t_{i+1}\}$ is limited by the current pair $\{s_i, t_i\}$ as follows:

- If $s_i \in F \Leftrightarrow t_i \notin F$, then the play terminates immediately, and player $\perp$ is declared the winner of the play.
- If the pair $\{s_i, t_i\}$ has been visited before in the play, i.e., there is some $j < i$ s.t. $\{s_i, t_i\} = \{s_j, t_j\}$, then the play terminates and player $\top$ wins; otherwise:
- Two tokens are put on the states $\{s_i, t_i\}$. If $s_i = t_i$, then both tokens lie on the same state.

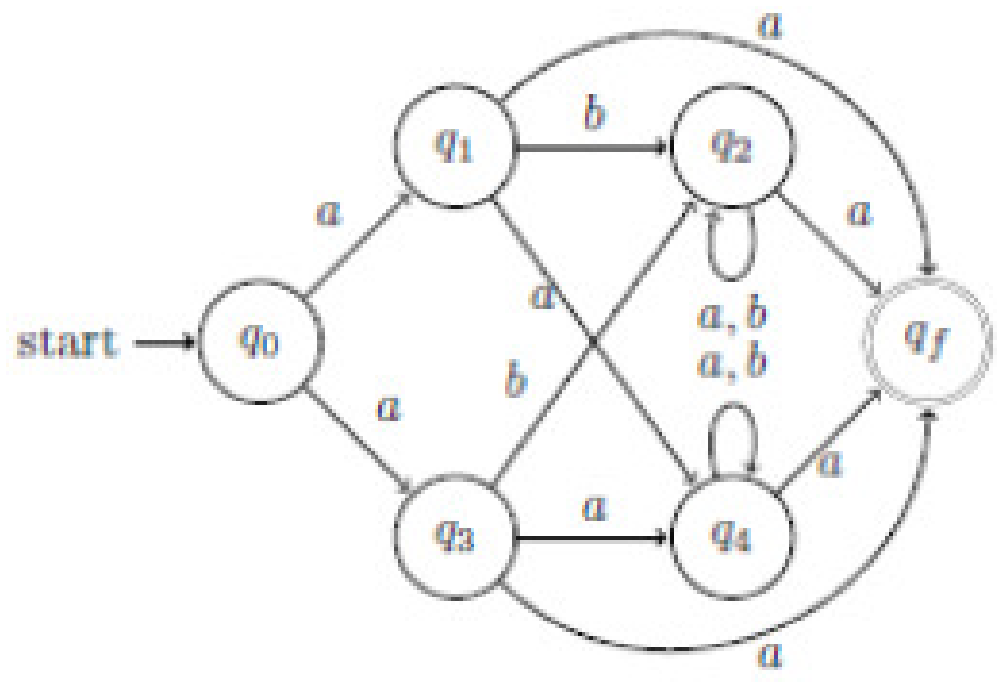
  Player $\perp$ then moves exactly one of the two tokens along an outgoing transition of the state the chosen token is located on.

  If both states do not have outgoing transitions, i.e., if player $\perp$ cannot move, player $\top$ wins; otherwise:
- Let a be the label of the transition along which player $\perp$ has moved.

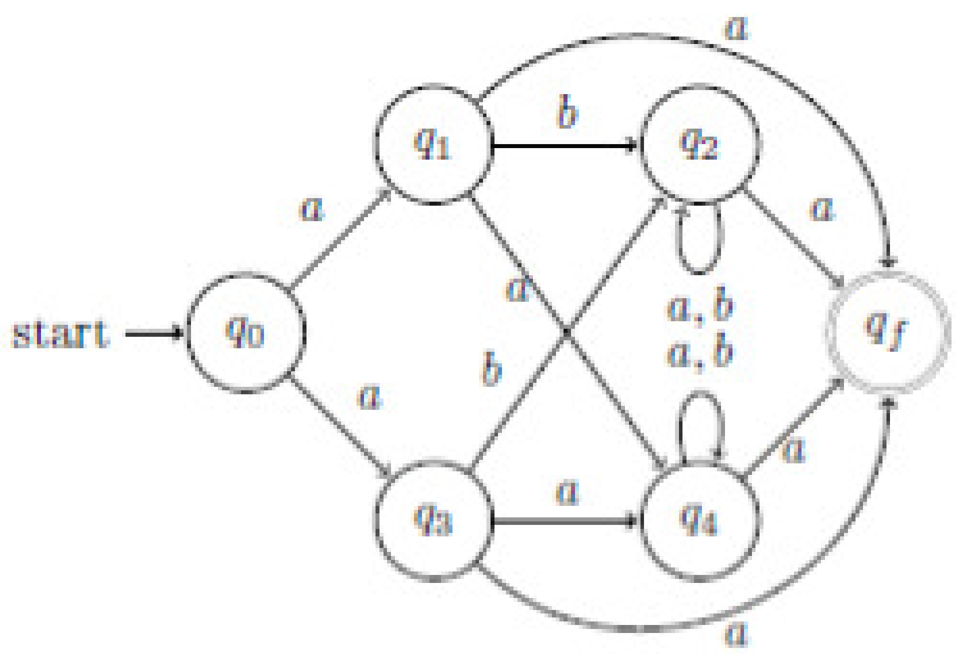
  Player $\top$ now has to try to match this move by moving the other token, i.e., the one that hasn't been moved in this round yet, along an outgoing transition also labeled by a.

  If player $\top$ cannot move in such a way, he immediately loses; otherwise:
- The new pair $\{s_{i+1}, t_{i+1}\}$ is determined by the states on which the two tokens are located.

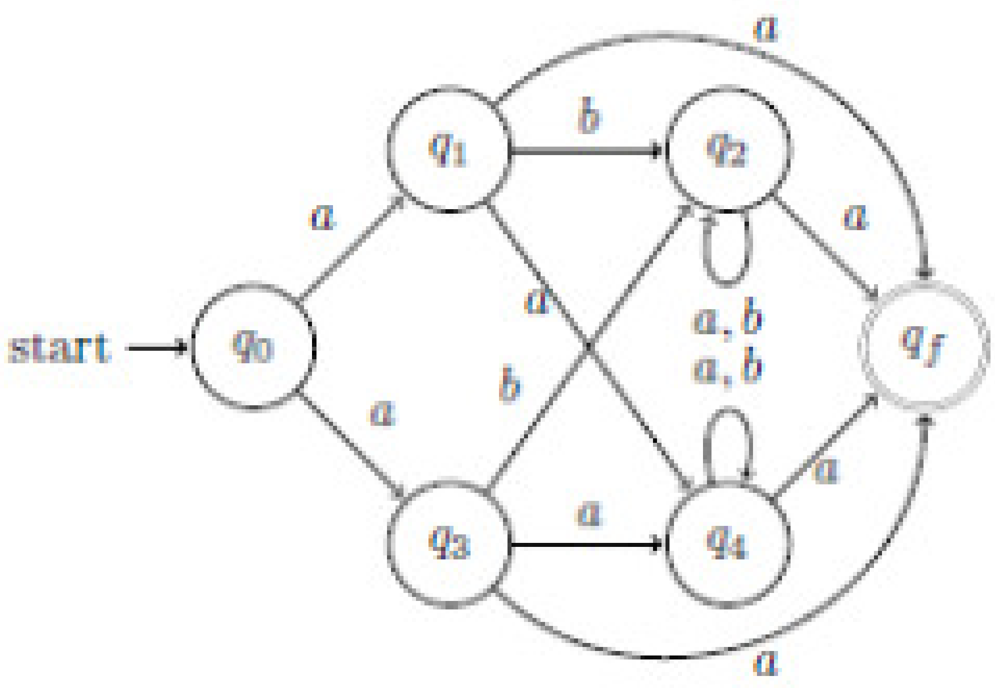
A player wins a pair $\{s_0, t_0\}$ of states if he can choose his moves in such a way that he wins any resulting play starting in $\{s_0, t_0\}$.



$\{q_4, q_f\}$



$\{q_1, q_2\}$



$\{q_0, q_3\}$

Q: If $p \sim q$ then $\top$ can win $\{p, q\}$?

↳ $\top$ can use the following "strategy":

choose move s.t. resulting pair is bisimilar again.

⌈Recall:

$p \sim q$ iff $p \in F \Leftrightarrow q \in F$

$\bot$ moves "green" token

$$\land\ \forall a \in \Sigma:\ \underbrace{\forall p' \in \delta(p,a)}\ \exists q' \in \delta(q,a): p' \sim q' \ \land\ \underbrace{\forall q' \in \delta(q,a)}\ \exists p' \in \delta(p,a): p' \sim q'$$

$\bot$ moves "red" token

$\top$ can still move to some bisimilar state

Q: If $p \sim q$ then $\top$ can win $\{p, q\}$?

$\hookrightarrow$ $\top$ can use the following "strategy":

choose move s.t. resulting pair is bisimilar again.

$\hookrightarrow$ Remaining possibilities:

a) $\bot$ cannot move

$\hookrightarrow$ as $p \sim q$ we have $p \in \overline{F} \Leftrightarrow q \in \overline{F}$

$\hookrightarrow$ $\top$ wins

go on $\leftarrow$ b) $\bot$ can move $\leadsto$ new bisimilar pair $p' \sim q'$

$\top$'s strategy

c) play visits a previously visited pair

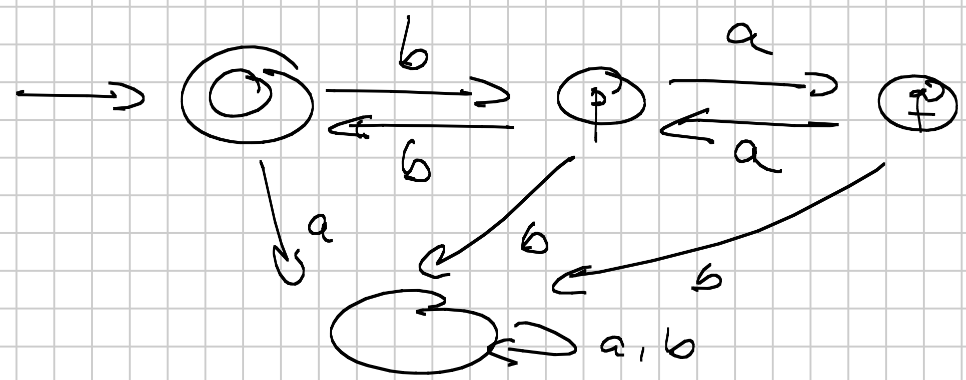
$\hookrightarrow$ $\top$ wins

Q: If $p \not\sim q$ then $\perp$ (can) win the pair $\{p, q\}$?

<u>First idea:</u> $\perp$ tries to play to non bisimilar pair.

$\leadsto$ not enough:

<u>DFA:</u>



$\leadsto$ possible play: $\{p, q\} \xrightarrow[\top: q, a]{\perp: p, a} \{p, q\}$

$p \not\sim q$

again $p \not\sim q$

but $\top$ wins!

Q: If $p \not\sim q$ then I (can) win the pair $\{p, q\}$?!

I's strategy: force play to pair $p \in F \Longleftrightarrow q \notin F$

along non-bisimilar pairs!

Recall:

Bisim(A)

Input: NFA $A = (Q, \Sigma, \delta, q_0, F)$;

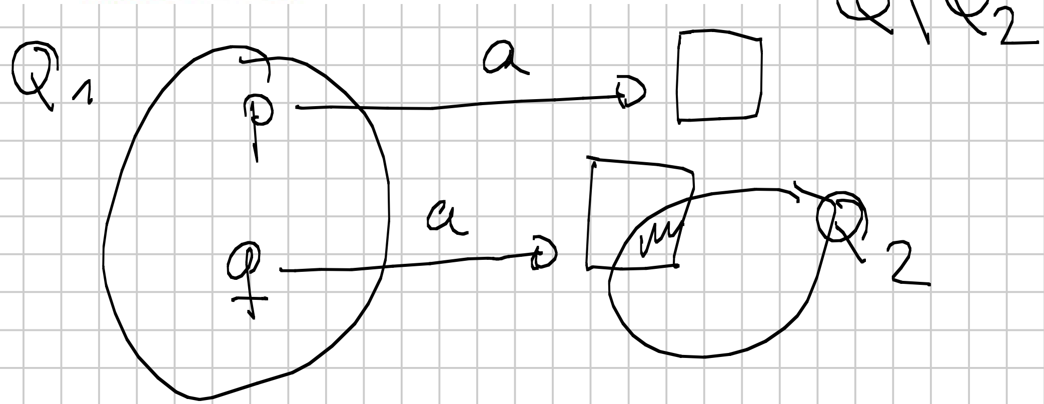
Output: the bisimilarity relation $\sim$;

$R \leftarrow \{F, Q \setminus F\}$

while *R is unstable* **do**

  choose $Q_1, Q_2 \in [Q]_R$ and $a \in \Sigma$ s.t. (a, Q_2) splits Q_1

  $R \leftarrow Ref_R[Q_1, a, Q_2]$



Q: If $p \not\sim q$ then $\perp$ (can) win the pair $\{p, q\}$?

Bisim(A)

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Output: the bisimilarity relation $\sim$;

1 $R \leftarrow \{F, Q \setminus F\}$

2 **while** *R is unstable* **do**

3 choose $Q_1, Q_2 \in [Q]_R$ and $a \in \Sigma$ s.t. (a, Q_2) splits Q_1

4 $R \leftarrow Ref_R[Q_1, a, Q_2]$

$\leftarrow$ Construct finite sequence of partitions of Q:

$\sim_0, \sim_1, \sim_2, \dots, \sim_e$

with $Q/\sim_0 = \{F, Q \setminus F\}$, $\sim_e = \sim$

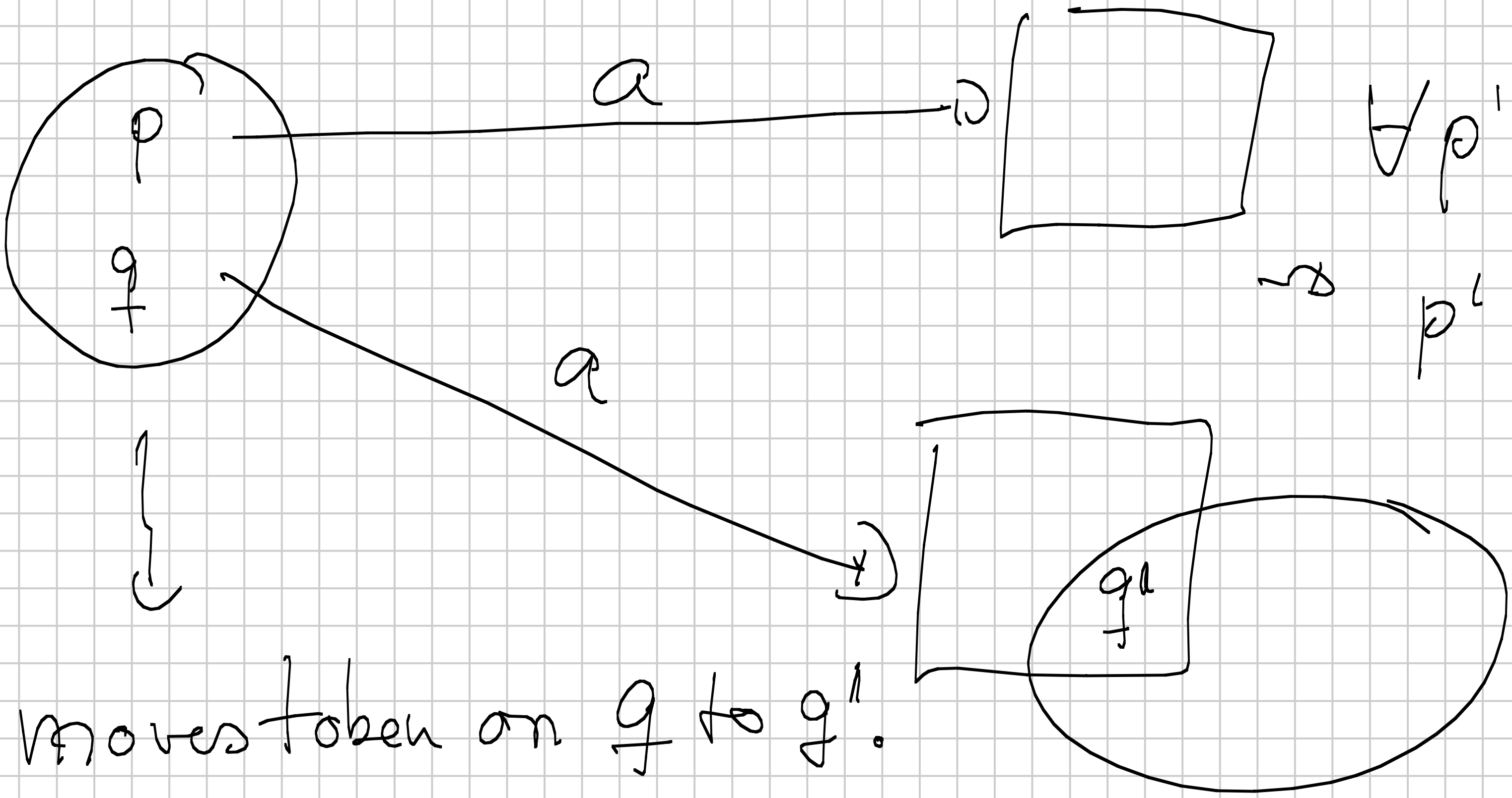
. For every pair $p \not\sim q$ there is a <u>smallest</u> k s.t.

$p \not\sim_k q$

Call this the <u>rank</u> of $\{p, q\}$

- If $k = \mathrm{rank}(\{p, q\})$ then $p \sim_{k-1} q$

$\rightsquigarrow$ $Q/_{\sim_{k-1}}$ p, q get separated by k^{th} split:



$\forall p' \in \delta(p, a)$:

$\rightsquigarrow$ $p' \not\sim_{k-1} q'$

I moves token on q to q'.

$\rightsquigarrow$ $\bot$'s strategy : if $p \not\sim q$ then <u>decrease the rank</u>

$\hookrightarrow$ all visited pairs are non-bisimilar

$\hookrightarrow$ So, if $\bot$ cannot move,

he wins as $p \in F \Leftrightarrow q \notin F$ has to hold

$\hookrightarrow$ Or the rank of the pair decreases

$\hookrightarrow$ eventually the play visits a pair $p \not\sim_0 q$

$\rightsquigarrow$ $\bot$ wins