

# COMPUTING THE LEAST FIXED POINT OF POSITIVE POLYNOMIAL SYSTEMS \*

JAVIER ESPARZA<sup>†</sup>, STEFAN KIEFER<sup>†</sup>, AND MICHAEL LUTTENBERGER<sup>†</sup>

**Abstract.** We consider equation systems of the form  $X_1 = f_1(X_1, \dots, X_n), \dots, X_n = f_n(X_1, \dots, X_n)$  where  $f_1, \dots, f_n$  are polynomials with *positive* real coefficients. In vector form we denote such an equation system by  $\mathbf{X} = \mathbf{f}(\mathbf{X})$  and call  $\mathbf{f}$  a system of positive polynomials, short SPP. Equation systems of this kind appear naturally in the analysis of stochastic models like stochastic context-free grammars (with numerous applications to natural language processing and computational biology), probabilistic programs with procedures, web-surfing models with back buttons, and branching processes. The least nonnegative solution  $\mu\mathbf{f}$  of an SPP equation  $\mathbf{X} = \mathbf{f}(\mathbf{X})$  is of central interest for these models. Etessami and Yannakakis [14] have suggested a particular version of Newton's method to approximate  $\mu\mathbf{f}$ .

We extend a result of Etessami and Yannakakis and show that Newton's method starting at  $\mathbf{0}$  always converges to  $\mu\mathbf{f}$ . We obtain lower bounds on the convergence speed of Newton's method and prove that for so-called *strongly connected* SPPs there exists a threshold  $k_{\mathbf{f}}$  such that for every  $i \geq 0$  the  $(k_{\mathbf{f}} + i)$ -th iteration of Newton's method has at least  $i$  valid bits of  $\mu\mathbf{f}$ . We also provide concrete bounds on  $k_{\mathbf{f}}$ . Further we show that Newton's method for arbitrary SPP equations also converges linearly, albeit the convergence rate, i.e., the number of bits obtained per iteration, is poorer. We prove bounds on the convergence rate, which we show to be essentially tight. Finally, we also provide a geometric interpretation of Newton's method for SPPs.

**Key words.** Newton's Method, Fixed-Point Equations, Polynomial Equations, Numerical Analysis, Convergence Speed

**AMS subject classifications.** 41A25, 65H10, 68Q60

**1. Introduction.** We consider equation systems of the form

$$\begin{aligned} X_1 &= f_1(X_1, \dots, X_n) \\ &\vdots \\ X_n &= f_n(X_1, \dots, X_n) \end{aligned}$$

where  $f_1, \dots, f_n$  are polynomials with *positive* real coefficients. In vector form we denote such an equation system by  $\mathbf{X} = \mathbf{f}(\mathbf{X})$ . The vector  $\mathbf{f}$  of polynomials is called a *system of positive polynomials*, or *SPP* for short.

Equation systems of this kind appear naturally in the analysis of context-free grammars (with numerous applications to natural language processing [25, 17] and computational biology [30, 7, 6, 22]), probabilistic programs with procedures [9, 4, 14, 11, 10, 12, 13], and web-surfing models with back buttons [15, 16]. More generally, they play an important rôle in the theory of *branching processes* [19, 2], stochastic processes describing the evolution of a population whose individuals can die and reproduce. The probability of extinction of the population is the least solution of such a system, a result whose history goes back to [31].

Since SPPs have positive coefficients,  $\mathbf{x} \leq \mathbf{x}'$  implies  $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{x}')$  for  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}_{\geq 0}^n$ , i.e., the functions  $f_1, \dots, f_n$  are monotonic. This allows us to apply Kleene's theorem (see for instance [23]), and conclude that a feasible system  $\mathbf{X} = \mathbf{f}(\mathbf{X})$ , i.e., one having at least one solution, has a smallest solution  $\mu\mathbf{f}$ . It follows easily from

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<sup>†</sup>Institut für Informatik, Technische Universität München, 85748 Garching, Germany, {esparza,kiefer,luttenbe}@model.in.tum.de

standard Galois theory that  $\mu\mathbf{f}$  can be irrational and non-expressible by radicals. The problem of deciding, given an SPP and a vector  $\mathbf{v}$  encoded in binary, whether  $\mu\mathbf{f} \leq \mathbf{v}$  holds, is known to be in PSPACE and to be at least as hard as SQUARE-ROOT-SUM, a well-known problem of computational geometry, whose membership in NP is a long standing open question, and at least as hard as PosSLP, the problem of deciding, giving a division-free straight-line program, whether it produces a positive integer (see [14] for more details). PosSLP has been recently shown to play a central rôle in understanding the Blum-Shub-Smale model of computation, where each single arithmetic operation over the reals can be carried out exactly and in constant time [1].

While these results provide important information on the complexity of solving SPP equations, for the practical applications mentioned above the problem of determining if  $\mu\mathbf{f}$  exceeds a given bound is less relevant than the complexity of, given a number  $i \geq 0$ , computing  $i$  valid bits of  $\mu\mathbf{f}$ , i.e., computing a vector  $\mathbf{v}$  such that  $|\mu\mathbf{f}_j - v_j| / |\mu\mathbf{f}_j| \leq 2^{-i}$  for every  $1 \leq j \leq n$ . In this paper we study this problem in the Blum-Shub-Smale model.

Since the least fixed point of a feasible SPP  $\mathbf{f}$  is a solution of  $\mathbf{F}(\mathbf{X}) = \mathbf{0}$  for  $\mathbf{F}(\mathbf{X}) = \mathbf{f}(\mathbf{X}) - \mathbf{X}$ , we can try to apply (the multivariate version of) *Newton's method* [27]: starting at some  $\mathbf{x}^{(0)} \in \mathbb{R}^n$  (we use uppercase to denote variables and lowercase to denote values), compute the sequence

$$\mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} - (\mathbf{F}'(\mathbf{x}^{(k)}))^{-1} \mathbf{F}(\mathbf{x}^{(k)})$$

where  $\mathbf{F}'(\mathbf{X})$  is the Jacobian matrix of partial derivatives. A first difficulty is that the method might not even be well-defined, because  $\mathbf{F}'(\mathbf{x}^{(k)})$  could be singular for some  $k$ . However, Etessami and Yannakakis have recently studied SPPs derived from probabilistic pushdown automata (actually, from an equivalent model called recursive Markov chains) [14], and shown that a particular version of Newton's method always converges, namely a version which decomposes the SPP into *strongly connected components* (SCCs)<sup>1</sup> and applies Newton's method to them in a bottom-up fashion. Our first result generalizes Etessami and Yannakakis': the ordinary Newton method converges for arbitrary SPPs, provided that they are clean (which can be easily achieved).

While these results show that Newton's method can be an adequate algorithm for solving SPP equations, they provide no information on the number of iterations needed to compute  $i$  valid bits. To the best of our knowledge (and perhaps surprisingly), the rest of the literature does not contain relevant information either: it has not considered SPPs explicitly, and the existing results have very limited interest for SPPs, since they do not apply even for very simple and relevant SPP cases (see *Related work* below).

We obtain upper bounds on the number of iterations that Newton's method needs to produce  $i$  valid bits, first for strongly connected and then for arbitrary SPP equations. Since one iteration requires  $\mathcal{O}(n^3)$  arithmetic operations in a system of  $n$  equations, we immediately obtain an upper bound on the time complexity of Newton's method in the Blum-Shub-Smale model. We prove that for strongly connected SPP equations  $\mathbf{X} = \mathbf{f}(\mathbf{X})$  there exists a threshold  $k_{\mathbf{f}}$  such that for every  $i \geq 0$  the  $(k_{\mathbf{f}} + i)$ -th iteration of Newton's method has at least  $i$  valid bits of  $\mu\mathbf{f}$ . So, loosely speaking, after  $k_{\mathbf{f}}$  iterations Newton's method is guaranteed to compute at least 1 new bit of the solution per iteration; we say that Newton's method converges at least

<sup>1</sup>Loosely speaking, a subset of variables and their associated equations form an SCC, if the value of any variable in the subset influences the value of all variables in the subset, see § 2 for details.

linearly with rate 1. Moreover, we show that the threshold  $k_{\mathbf{f}}$  can be chosen as

$$k_{\mathbf{f}} = \lceil 4mn + 3n \max\{0, -\log \mu_{\min}\} \rceil$$

where  $n$  is the number of polynomials of the SPP,  $m$  is such that all coefficients of the SPP can be given as ratios of  $m$ -bit integers, and  $\mu_{\min}$  is the minimal component of the least fixed point  $\mu\mathbf{f}$ .

Notice that  $k_{\mathbf{f}}$  depends on  $\mu\mathbf{f}$ , which is what Newton's method should compute. For this reason we also obtain bounds on  $k_{\mathbf{f}}$  depending only on  $m$  and  $n$ . We show that for arbitrary strongly connected SPP equations  $k_{\mathbf{f}} = 4mn2^n$  is also a valid threshold. For SPP equations coming from stochastic models, such as the ones listed above, we do far better. First, we show that if every procedure has a non-zero probability of terminating (a condition that always holds for back-button processes [15, 16]), then a valid threshold is  $k_{\mathbf{f}} = 2m(n+1)$ . As a corollary, our result shows that for back-button processes,  $i$  valid bits can be computed in time  $\mathcal{O}(mn^4 + in^3)$  in the Blum-Shub-Smale model. Second, we observe that, since  $\mathbf{x}^{(k)} \leq \mathbf{x}^{(k+1)} \leq \mu\mathbf{f}$  holds for every  $k \geq 0$ , as Newton's method proceeds it provides better and better lower bounds for  $\mu_{\min}$  and thus for  $k_{\mathbf{f}}$ . We exhibit an SPP for which, using this fact and our theorem, we can prove that no component of the solution reaches the value 1. This cannot be proved by just computing more iterations, no matter how many.

For general SPP equations, not necessarily strongly connected, we show that Newton's method still converges linearly, albeit the convergence rate is poorer. We exhibit a family of SPPs showing that this bound is essentially tight.

Finally, the last result of the paper concerns the geometric interpretation of Newton's method for SPP equations. We show that, loosely speaking, the Newton approximants stay within the hypervolume limited by the hypersurfaces corresponding to each individual equation. This means that a simple geometric intuition of how Newton's method works, extracted from the case of 2-dimensional SPPs, is also correct for arbitrary dimensions. Our result also leads to a new variant of Newton's method.

*Related work.* There is a large body of literature on the convergence speed of Newton's method for arbitrary systems of differentiable functions. A comprehensive reference is Ortega and Rheinboldt's book [27] (see also Chapter 8 of Ortega's course [26] or Chapter 5 of [20] for a brief summary). Several theorems (for instance Theorem 8.1.10 of [26]) prove that the number of valid bits grows linearly, superlinearly, or even exponentially in the number of iterations, but only under the hypothesis that  $\mathbf{F}'(\mathbf{x})$  is non-singular everywhere, in a neighborhood of  $\mu\mathbf{f}$ , or at least at the point  $\mu\mathbf{f}$  itself. However, the matrix  $\mathbf{F}'(\mu\mathbf{f})$  can be singular for an SPP, even for the 1-dimensional SPP  $f(X) = 1/2X^2 + 1/2$ .

The general case in which  $\mathbf{f}'(\mu\mathbf{f})$  may be singular for the solution  $\mu\mathbf{f}$  the method converges to has been thoroughly studied. In a seminal paper [29], Reddien shows that under certain conditions, the main ones being that the kernel of  $\mathbf{f}'(\mu\mathbf{f})$  has dimension 1 and that the initial point is close enough to the solution, Newton's method gains 1 bit per iteration. Decker and Kelly obtain results for kernels of arbitrary dimension, but they require a certain linear map  $B(\mathbf{X})$  to be non-singular for all  $\mathbf{x} \neq \mathbf{0}$  [5]. Griewank observes in [18] that the non-singularity of  $B(\mathbf{X})$  is in fact a strong condition which, in particular, can only be satisfied by kernels of even dimension. He presents a weaker sufficient condition for linear convergence requiring  $B(\mathbf{X})$  to be non-singular only at the initial point  $\mathbf{x}^{(0)}$ , i.e., it only requires to make "the right guess" for  $\mathbf{x}^{(0)}$ . Unfortunately, none of these results can be directly applied to arbitrary SPPs. The

possible dimensions of the kernel of  $\mathbf{f}'(\mu\mathbf{f})$  for an SPP  $\mathbf{f}(\mathbf{X})$  are to the best of our knowledge unknown, and deciding this question seems as hard as those related to the convergence rate<sup>2</sup>. Griewank's result does not apply to the decomposed Newton's method either because the mapping  $B(\mathbf{x}^{(0)})$  is always singular for  $\mathbf{x}^{(0)} = \mathbf{0}$ .

Kantorovich's famous theorem (see e.g. Theorem 8.2.6 of [27] and [28] for an improvement) guarantees global convergence and only requires  $\mathbf{F}'$  to be non-singular at  $\mathbf{x}^{(0)}$ . However, it also requires to find a Lipschitz constant for  $\mathbf{f}'$  on a suitable region and some other bounds on  $\mathbf{f}'$ . These latter conditions are far too restrictive for the applications mentioned above. For instance, the stochastic context-free grammars whose associated SPPs satisfy Kantorovich's conditions cannot exhibit two productions  $X \rightarrow aYZ$  and  $W \rightarrow \varepsilon$  such that  $\text{Prob}(X \rightarrow aYZ) \cdot \text{Prob}(W \rightarrow \varepsilon) \geq 1/4$ . This class of grammars is too contrived to be of use.

Summarizing, while the convergence of Newton's method for systems of differentiable functions has been intensely studied, the case of SPPs does not seem to have been considered yet. The results obtained for other classes have very limited applicability to SPPs: either they do not apply at all, or only apply to contrived SPP subclasses. Moreover, these results only provide information about the growth rate of the number of accurate bits, but not about the number itself. Our thresholds lead to *explicit* upper bounds for the number of accurate bits depending only on syntactical parameters: the number of equations and the size of the coefficients.

*Structure of the paper.* § 2 defines SPPs and briefly describes their applications to stochastic systems. § 3 presents a short summary of our main theorems. § 4 proves some fundamental properties of Newton's method for SPP equations. § 5 and § 6 contain our results on the convergence speed for strongly connected and general SPP equations, respectively. § 7 shows that the bounds are essentially tight. § 8 presents our results about the geometrical interpretation of Newton's method, and § 9 contains conclusions.

**2. Preliminaries.** In this section we introduce our notation used in the following and formalize the concepts mentioned in the introduction.

**2.1. Notation.** As usual,  $\mathbb{R}$  and  $\mathbb{N}$  denote the set of real, respectively natural numbers. We assume  $0 \in \mathbb{N}$ .  $\mathbb{R}^n$  denotes the set of  $n$ -dimensional real valued column vectors and  $\mathbb{R}_{\geq 0}^n$  the subset of vectors with nonnegative components. We use bold letters for vectors, e.g.  $\mathbf{x} \in \mathbb{R}^n$ , where we assume that  $\mathbf{x}$  has the components  $x_1, \dots, x_n$ . Similarly, the  $i$ -th component of a function  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is denoted by  $f_i$ . We define  $\mathbf{0} := (0, \dots, 0)^\top$  and  $\mathbf{1} := (1, \dots, 1)^\top$  where the superscript  $\top$  indicates the transpose of a vector or a matrix. Let  $\|\cdot\|$  denote some norm on  $\mathbb{R}^n$ . Sometimes we use explicitly the maximum norm  $\|\cdot\|_\infty$  with  $\|\mathbf{x}\|_\infty := \max_{1 \leq i \leq n} |x_i|$ .

The partial order  $\leq$  on  $\mathbb{R}^n$  is defined as usual by setting  $\mathbf{x} \leq \mathbf{y}$  if  $x_i \leq y_i$  for all  $1 \leq i \leq n$ . Similarly,  $\mathbf{x} < \mathbf{y}$  if  $\mathbf{x} \leq \mathbf{y}$  and  $\mathbf{x} \neq \mathbf{y}$ . Finally, we write  $\mathbf{x} \prec \mathbf{y}$  if  $x_i < y_i$  for all  $1 \leq i \leq n$ , i.e., if every component of  $\mathbf{x}$  is smaller than the corresponding component of  $\mathbf{y}$ .

We use  $X_1, \dots, X_n$  as variable identifiers and arrange them into the vector  $\mathbf{X}$ . In the following  $n$  always denotes the number of variables, i.e., the dimension of  $\mathbf{X}$ . While  $\mathbf{x}, \mathbf{y}, \dots$  denote arbitrary elements in  $\mathbb{R}^n$ , resp.  $\mathbb{R}_{\geq 0}^n$ , we write  $\mathbf{X}$  if we want to emphasize that a function is given w.r.t. these variables. Hence,  $\mathbf{f}(\mathbf{X})$  represents the function itself, whereas  $\mathbf{f}(\mathbf{x})$  denotes its value for some  $\mathbf{x} \in \mathbb{R}^n$ .

<sup>2</sup>More precisely, SPPs with kernels of arbitrary dimension exist, but the cases we know of can be trivially reduced to SPPs with kernels of dimension 1.

If  $S \subseteq \{1, \dots, n\}$  is a set of components and  $\mathbf{x}$  a vector, then by  $\mathbf{x}_S$  we mean the vector obtained by restricting  $\mathbf{x}$  to the components in  $S$ .

Let  $S \subseteq \{1, \dots, n\}$  and  $\bar{S} = \{1, \dots, n\} \setminus S$ . Given a function  $\mathbf{f}(\mathbf{X})$  and a vector  $\mathbf{x}_S$ , then  $\mathbf{f}[S/\mathbf{x}_S]$  is obtained by replacing, for each  $s \in S$ , each occurrence of  $\mathbf{X}_s$  by  $\mathbf{x}_s$  and removing the  $s$ -component. In other words, if  $\mathbf{f}(\mathbf{X}) = \mathbf{f}(\mathbf{X}_S, \mathbf{X}_{\bar{S}})$  then  $\mathbf{f}[S/\mathbf{x}_S](\mathbf{y}_{\bar{S}}) = \mathbf{f}_{\bar{S}}(\mathbf{x}_S, \mathbf{y}_{\bar{S}})$ . For instance, if  $\mathbf{f}(X_1, X_2) = (X_1 X_2 + \frac{1}{2}, X_2^2 + \frac{1}{5})^\top$ , then  $\mathbf{f}[\{2\}/\frac{1}{2}] : \mathbb{R} \rightarrow \mathbb{R}, X_1 \mapsto \frac{1}{2}X_1 + \frac{1}{2}$ .

$\mathbb{R}^{m \times n}$  denotes the set of matrices having  $m$  rows and  $n$  columns. The transpose of a vector or matrix is indicated by the superscript  $^\top$ . The identity matrix of  $\mathbb{R}^{n \times n}$  is denoted by  $\text{Id}$ .

The *formal Neumann series* of  $A \in \mathbb{R}^{n \times n}$  is defined by  $A^* = \sum_{k \in \mathbb{N}} A^k$ . It is well-known that  $A^*$  exists if and only if the spectral radius of  $A$  is less than 1, i.e.  $\max\{|\lambda| \mid \mathbb{C} \ni \lambda \text{ is an eigenvalue of } A\} < 1$ . If  $A^*$  exists then  $A^* = (\text{Id} - A)^{-1}$ .

The partial derivative of a function  $f(\mathbf{X}) : \mathbb{R}^n \rightarrow \mathbb{R}$  w.r.t. the variable  $X_i$  is denoted by  $\partial_{X_i} f$ . The gradient  $\nabla f$  of  $f(\mathbf{X})$  is then defined to be the (row) vector

$$\nabla f := (\partial_{X_1} f, \dots, \partial_{X_n} f).$$

The *Jacobian* of a function  $\mathbf{f}(\mathbf{X})$  with  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the matrix  $\mathbf{f}'(\mathbf{X})$  defined by

$$\mathbf{f}'(\mathbf{X}) = \begin{pmatrix} \partial_{X_1} f_1 & \dots & \partial_{X_n} f_1 \\ \vdots & & \vdots \\ \partial_{X_1} f_m & \dots & \partial_{X_n} f_m \end{pmatrix},$$

i.e., the  $i$ -th row of  $\mathbf{f}'$  is the gradient of  $f_i$ .

## 2.2. Systems of Positive Polynomials.

**DEFINITION 2.1.** A function  $\mathbf{f}(\mathbf{X})$  with  $\mathbf{f} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$  is a system of positive polynomials (SPP), if every component  $f_i(\mathbf{X})$  is a polynomial in the variables  $X_1, \dots, X_n$  with coefficients in  $\mathbb{R}_{\geq 0}$ . We call an SPP  $\mathbf{f}(\mathbf{X})$  feasible if  $\mathbf{y} = \mathbf{f}(\mathbf{y})$  for some  $\mathbf{y} \in \mathbb{R}_{\geq 0}^n$ . An SPP is called linear (resp. quadratic) if all polynomials have degree at most 1 (resp. 2).

**FACT 2.2.** Every SPP  $\mathbf{f}$  is monotone on  $\mathbb{R}_{\geq 0}^n$ , i.e. for  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$  we have  $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{y})$ .

We will need the following lemma, a version of Taylor's theorem.

**LEMMA 2.3 (Taylor).** Let  $\mathbf{f}$  be an SPP and  $\mathbf{x}, \mathbf{u} \geq \mathbf{0}$ . Then

$$\mathbf{f}(\mathbf{x}) + \mathbf{f}'(\mathbf{x})\mathbf{u} \leq \mathbf{f}(\mathbf{x} + \mathbf{u}) \leq \mathbf{f}(\mathbf{x}) + \mathbf{f}'(\mathbf{x} + \mathbf{u})\mathbf{u}.$$

*Proof.* It suffices to show this for a multivariate polynomial  $f(\mathbf{X})$  with nonnegative coefficients. Consider  $g(t) = f(\mathbf{x} + t\mathbf{u})$ . We then have

$$f(\mathbf{x} + \mathbf{u}) = g(1) = g(0) + \int_0^1 g'(s) ds = f(\mathbf{x}) + \int_0^1 f'(\mathbf{x} + s\mathbf{u})\mathbf{u} ds.$$

The result follows as  $f'(\mathbf{x}) \leq f'(\mathbf{x} + s\mathbf{u}) \leq f'(\mathbf{x} + \mathbf{u})$  for  $s \in [0, 1]$ .  $\square$

Since every SPP is continuous, Kleene's fixed-point theorem (see e.g. [23]) applies.

**THEOREM 2.4 (Kleene's fixed-point theorem).** Every feasible SPP  $\mathbf{f}$  has a least fixed point  $\mu\mathbf{f}$  in  $\mathbb{R}_{\geq 0}^n$  i.e.,  $\mu\mathbf{f} = \mathbf{f}(\mu\mathbf{f})$  and, in addition,  $\mathbf{y} = \mathbf{f}(\mathbf{y})$  implies  $\mu\mathbf{f} \leq \mathbf{y}$ .

Moreover, the sequence  $(\kappa_{\mathbf{f}}^{(k)})_{k \in \mathbb{N}}$  with  $\kappa_{\mathbf{f}}^{(k)} = \mathbf{f}^k(\mathbf{0})$  is monotonically increasing with respect to  $\leq$  (i.e.  $\kappa_{\mathbf{f}}^{(k)} \leq \kappa_{\mathbf{f}}^{(k+1)}$ ) and converges to  $\mu\mathbf{f}$ .

In the following we call  $(\kappa_{\mathbf{f}}^{(k)})_{k \in \mathbb{N}}$  the *Kleene sequence* of  $\mathbf{f}(\mathbf{X})$ , and drop the subscript whenever  $\mathbf{f}$  is clear from the context. Similarly, we sometimes write  $\mu$  instead of  $\mu\mathbf{f}$ .

An SPP  $\mathbf{f}(\mathbf{X})$  is *clean* if for all variables  $X_i$  there is a  $k \in \mathbb{N}$  such that  $\kappa_i^{(k)} > 0$ . It is easy to see that we have  $\kappa_i^{(k)} = 0$  for all  $k \in \mathbb{N}$  if  $\kappa_i^{(n)} = 0$ . So we can “clean” an SPP  $\mathbf{f}(\mathbf{X})$  in time linear in the size of  $\mathbf{f}$  by determining the components  $i$  with  $\kappa_i^{(n)} = 0$  and removing them.

NOTATION 2.5. *In the following, we always assume that an SPP  $\mathbf{f}$  is clean and feasible. I.e., whenever we write “SPP”, we mean “clean and feasible SPP”, unless explicitly stated otherwise.*

We will also need the notion of *dependence* between variables.

DEFINITION 2.6. *Let  $f(\mathbf{X})$  be a polynomial. We say,  $f(\mathbf{X})$  contains a variable  $X_i$  if  $\partial_{X_i} f(\mathbf{X})$  is not the zero-polynomial.*

DEFINITION 2.7. *Let  $\mathbf{f}(\mathbf{X})$  be an SPP. A component  $i$  depends directly on a component  $k$  if  $f_i(\mathbf{X})$  contains  $X_k$ . A component  $i$  depends on  $k$  if either  $i$  depends directly on  $k$  or there is a component  $j$  such that  $i$  depends on  $j$  and  $j$  depends on  $k$ . The components  $\{1, \dots, n\}$  can be partitioned into strongly components (SCCs) where an SCC  $S$  is a maximal set of components such that each component in  $S$  depends on each other component in  $S$ . An SCC is called trivial if it consists of a single component that does not depend on itself. An SPP is strongly connected (short: an scSPP) if  $\{1, \dots, n\}$  is a non-trivial SCC.*

**2.3. Convergence Speed.** We will analyze the convergence speed of Newton’s method. To this end we need the notion of *valid bits*.

DEFINITION 2.8. *Let  $\mathbf{f}$  be an SPP. A vector  $\mathbf{x}$  has  $i$  valid bits of the least fixed point  $\mu\mathbf{f}$  if*

$$\frac{|\mu\mathbf{f}_j - x_j|}{|\mu\mathbf{f}_j|} \leq 2^{-i}$$

for every  $1 \leq j \leq n$ . Let  $(\mathbf{x}^{(k)})_{k \in \mathbb{N}}$  be a sequence with  $\mathbf{0} \leq \mathbf{x}^{(k)} \leq \mu\mathbf{f}$ . Then the convergence order  $\beta : \mathbb{N} \rightarrow \mathbb{N}$  of the sequence  $(\mathbf{x}^{(k)})_{k \in \mathbb{N}}$  is defined as follows:  $\beta(k)$  is the greatest natural number  $i$  such that  $\mathbf{x}^{(k)}$  has  $i$  valid bits (or  $\infty$  if such a greatest number does not exist). We will always mean the convergence order of the Newton sequence  $(\nu^{(k)})_{k \in \mathbb{N}}$ , unless explicitly stated otherwise.

We say that a sequence has linear, exponential, logarithmic, etc. convergence order if the function  $\beta(k)$  grows linearly, exponentially, or logarithmically in  $k$ , respectively.

**2.4. Stochastic Models.** As mentioned in the introduction, several problems concerning stochastic models can be reduced to problems about the least fixed point  $\mu\mathbf{f}$  of an SPP  $\mathbf{f}$ . In these cases,  $\mu\mathbf{f}$  is a vector of probabilities, and so  $\mu\mathbf{f} \leq \mathbf{1}$ .

**2.4.1. Probabilistic Pushdown Automata.** Our study of SPPs was initially motivated by the verification of probabilistic pushdown automata. A *probabilistic pushdown automaton* (pPDA) is a tuple  $\mathcal{P} = (Q, \Gamma, \delta, Prob)$  where  $Q$  is a finite set of control states,  $\Gamma$  is a finite stack alphabet,  $\delta \subseteq Q \times \Gamma \times Q \times \Gamma^*$  is a finite transition relation (we write  $pX \hookrightarrow q\alpha$  instead of  $(p, X, q, \alpha) \in \delta$ ), and  $Prob$  is a function which to each transition  $pX \hookrightarrow q\alpha$  assigns its probability  $Prob(pX \hookrightarrow q\alpha) \in (0, 1]$  so that

for all  $p \in Q$  and  $X \in \Gamma$  we have  $\sum_{pX \xrightarrow{x} q\alpha} \text{Prob}(pX \xrightarrow{x} q\alpha) = 1$ . We write  $pX \xrightarrow{x} q\alpha$  instead of  $\text{Prob}(pX \xrightarrow{x} q\alpha) = x$ . A *configuration* of  $\mathcal{P}$  is a pair  $qw$ , where  $q$  is a control state and  $w \in \Gamma^*$  is a *stack content*. A probabilistic pushdown automaton  $\mathcal{P}$  naturally induces a possibly infinite Markov chain with the configurations as states and transitions given by:  $pX\beta \xrightarrow{x} q\alpha\beta$  for every  $\beta \in \Gamma^*$  iff  $pX \xrightarrow{x} q\alpha$ . We assume w.l.o.g. that if  $pX \xrightarrow{x} q\alpha$  is a transition then  $|\alpha| \leq 2$ .

pPDAs and the equivalent model of recursive Markov chains have been very thoroughly studied [9, 4, 14, 11, 10, 12, 13]. This work has shown that the key to the analysis of pPDAs are the *termination probabilities*  $[pXq]$ , where  $p$  and  $q$  are states, and  $X$  is a stack letter, defined as follows (see e.g. [9] for a more formal definition):  $[pXq]$  is the probability that, starting at the configuration  $pX$ , the pPDA eventually reaches the configuration  $q\varepsilon$  (empty stack). It is not difficult to show that the vector of these probabilities is the least solution of the SPP equation system containing the equation

$$\langle pXq \rangle = \sum_{pX \xrightarrow{x} rYZ} x \cdot \sum_{t \in Q} \langle rYt \rangle \cdot \langle tZq \rangle + \sum_{pX \xrightarrow{x} rY} x \cdot \langle rYq \rangle + \sum_{pX \xrightarrow{x} q\varepsilon} x$$

for each triple  $(p, X, q)$ . Call this quadratic SPP the *termination SPP* of the pPDA (we assume that termination SPPs are clean, and it is easy to see that they are always feasible).

**2.4.2. Strict pPDAs and Back-Button Processes.** A pPDA is *strict* if for all  $pX \in Q \times \Gamma$  and all  $q \in Q$  the transition relation contains a pop-rule  $pX \xrightarrow{x} q\varepsilon$  for some  $x > 0$ . Essentially, strict pPDAs model programs in which every procedure has at least one terminating execution that does not call any other procedure. The termination SPP of a strict pPDA satisfies  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$ .

In [15, 16] a class of stochastic processes is introduced to model the behavior of web-surfers who from the current webpage  $A$  can decide either to follow a link to another page, say  $B$ , with probability  $\ell_{AB}$ , or to press the “back button” with nonzero probability  $b_A$ . These back-button processes correspond to a very special class of strict pPDAs having one single control state (which in the following we omit), and rules of the form  $A \xrightarrow{b_A} \varepsilon$  (press the back button from  $A$ ) or  $A \xrightarrow{\ell_{AB}} BA$  (follow the link from  $A$  to  $B$ , remembering  $A$  as destination of pressing the back button at  $B$ ). The termination probabilities are given by an SPP equation system containing the equation

$$\langle A \rangle = b_A + \sum_{A \xrightarrow{\ell_{AB}} BA} \ell_{AB} \langle B \rangle \langle A \rangle = b_A + \langle A \rangle \sum_{A \xrightarrow{\ell_{AB}} BA} \ell_{AB} \langle B \rangle$$

for every webpage  $A$ . In [15, 16] those termination probabilities are called *revocation* probabilities. The revocation probability of a page  $A$  is the probability that, when currently visiting webpage  $A$  and having  $H_0H_1 \dots H_{n-1}H_n$  as the browser history of previously visited pages, then during subsequent surfing from  $A$  the random user eventually returns to webpage  $H_n$  with  $H_0H_1 \dots H_{n-1}$  as the remaining browser history.

EXAMPLE 2.9. Consider the following equation system.

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0.4X_2X_1 + 0.6 \\ 0.3X_1X_2 + 0.4X_3X_2 + 0.3 \\ 0.3X_1X_3 + 0.7 \end{pmatrix}$$

The least solution of the system gives the revocation probabilities of a back-button process with three web-pages. For instance, if the surfer is at page 2 it can choose between following links to pages 1 and 3 with probabilities 0.3 and 0.4, respectively, or pressing the back button with probability 0.3.

**3. Newton's Method and an Overview of Our Results.** In order to approximate the least fixed point  $\mu\mathbf{f}$  of an SPP  $\mathbf{f}$  we employ Newton's method:

DEFINITION 3.1. Let  $\mathbf{f}$  be an SPP. The Newton operator  $\mathcal{N}_{\mathbf{f}}$  is defined as follows:

$$\mathcal{N}_{\mathbf{f}}(\mathbf{X}) := \mathbf{X} + (\text{Id} - \mathbf{f}'(\mathbf{X}))^{-1}(\mathbf{f}(\mathbf{X}) - \mathbf{X})$$

The sequence  $(\nu_{\mathbf{f}}^{(k)})_{k \in \mathbb{N}}$  with  $\nu_{\mathbf{f}}^{(k)} = \mathcal{N}_{\mathbf{f}}^k(\mathbf{0})$  is called Newton sequence. We drop the subscript of  $\mathcal{N}_{\mathbf{f}}$  and  $\nu_{\mathbf{f}}^{(k)}$  when  $\mathbf{f}$  is understood.

The main results of this paper concern the application of Newton's method to SPPs. We summarize them in this section.

**Theorem 4.8** states that the Newton sequence  $(\nu^{(k)})_{k \in \mathbb{N}}$  is well-defined (i.e., the inverse matrices  $(\text{Id} - \mathbf{f}'(\nu^{(k)}))^{-1}$  exist for every  $k \in \mathbb{N}$ ), monotonically increasing and bounded from above by  $\mu\mathbf{f}$  (i.e.  $\nu^{(k)} \leq \mathbf{f}(\nu^{(k)}) \leq \nu^{(k+1)} \leq \mu\mathbf{f}$ ), and converges to  $\mu\mathbf{f}$ . This theorem generalizes the result of Etessami and Yannakakis in [14] to arbitrary SPPs and to the ordinary Newton's method.

For more quantitative results on the convergence speed it is convenient to focus on quadratic SPPs. **Theorem 4.14** shows that any SPP can be syntactically transformed into a quadratic SPP without changing the least fixed point and without accelerating Newton's method. This means, one can perform Newton's method on the original (possibly non-quadratic) SPP and convergence will be at least as fast as for the corresponding quadratic SPP.

For quadratic SPPs, one iteration of Newton's method involves  $\mathcal{O}(n^3)$  arithmetical operations and  $\mathcal{O}(n^3)$  operations in the Blum-Shub-Smale model. Hence, any bound on the number of iterations needed to compute a given number of valid bits immediately leads to a bound on the number of operations. In § 5 we prove such bounds for *strongly connected* quadratic SPPs. We give different thresholds for the number of iterations, and show that when any of these thresholds is reached, Newton's method gains at least one valid bit for each iteration. More precisely, **Theorem 5.12** states the following. Let  $\mathbf{f}$  be a quadratic scSPP, let  $\mu_{\min}$  and  $\mu_{\max}$  be the minimal and maximal component of  $\mu\mathbf{f}$ , respectively, and let the coefficients of  $\mathbf{f}$  be given as ratios of  $m$ -bit integers. Then  $\beta(k_{\mathbf{f}} + i) \geq i$  holds for all  $i \in \mathbb{N}$  and for any of the following choices of  $k_{\mathbf{f}}$ :

1.  $4mn + \lceil 3n \max\{0, -\log \mu_{\min}\} \rceil$ ;
2.  $4mn2^n$ ;
3.  $7mn$  if  $\mathbf{f}$  satisfies  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$ ;
4.  $2m(n+1)$  if  $\mathbf{f}$  satisfies both  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$  and  $\mu_{\max} \leq 1$ .

We further show that Newton iteration can also be used to obtain a sequence of *upper* approximations of  $\mu\mathbf{f}$ . Those upper approximations converge to  $\mu\mathbf{f}$ , asymptotically as fast as the Newton sequence. More precisely, **Theorem 5.15** states the following: Let  $\mathbf{f}$  be a quadratic scSPP, let  $c_{\min}$  be the smallest nonzero coefficient of  $\mathbf{f}$ , and let  $\mu_{\min}$  be the minimal component of  $\mu\mathbf{f}$ . Further, for all Newton approximants



$\nu^{(k)}$  with  $\nu^{(k)} \succ \mathbf{0}$ , let  $\nu_{min}^{(k)}$  be the smallest coefficient of  $\nu^{(k)}$ . Then

$$\nu^{(k)} \leq \mu \mathbf{f} \leq \nu^{(k)} + \left[ \frac{\|\nu^{(k)} - \nu^{(k-1)}\|_\infty}{\left(c_{min} \cdot \min\{\nu_{min}^{(k)}, 1\}\right)^n} \right]$$

where  $[s]$  denotes the vector  $\mathbf{x}$  with  $x_j = s$  for all  $1 \leq j \leq n$ .

In § 6 we turn to general (not necessarily strongly connected) SPPs. We show in **Theorem 6.5** that Newton's method converges linearly and give a bound on the convergence rate, i.e., the number of iterations that is asymptotically needed to gain one valid bit. More precisely, the theorem proves that for every quadratic SPP  $\mathbf{f}$ , there is a threshold  $k_{\mathbf{f}} \in \mathbb{N}$  such that  $\beta(k_{\mathbf{f}} + i \cdot n \cdot 2^n) \geq i$  for all  $i \in \mathbb{N}$ . That is, in the worst case  $n \cdot 2^n$  extra iterations are needed in order to get one new valid bit. § 7 shows that the bound is essentially tight.

§ 8 gives a geometrical interpretation of Newton's method on quadratic SPP equations. Let  $R$  be the region bounded by the coordinate axes and by the quadrics corresponding to the individual equations. **Theorem 8.10** shows that all Kleene and Newton approximations lie within  $R$ , i.e.:  $\nu^{(i)}, \kappa^{(i)} \in R$  for every  $i \in \mathbb{N}$ .

#### 4. Fundamental Properties of Newton's Method.

**4.1. Effectiveness.** Eteessami and Yannakakis [14] suggested to use Newton's method for SPPs. More precisely, they showed that the sequence obtained by applying Newton's method to the equation system  $\mathbf{X} = \mathbf{f}(\mathbf{X})$  converges to  $\mu \mathbf{f}$  as long as  $\mathbf{f}$  is strongly connected. We extend their result to arbitrary SPPs, thereby reusing and extending several proofs of [14].

Recall Definition 2.8 where we defined the Newton operator  $\mathcal{N}_{\mathbf{f}}$  as

$$\mathcal{N}_{\mathbf{f}}(\mathbf{X}) := \mathbf{X} + (\text{Id} - \mathbf{f}'(\mathbf{X}))^{-1} (\mathbf{f}(\mathbf{X}) - \mathbf{X})$$

and the associated Newton sequence  $(\nu^{(k)})_{k \in \mathbb{N}}$ . Before we prove some fundamental properties of the Newton sequence (see Theorem 4.8 below) we study a sequence generated by a somewhat weaker version of the Newton operator:

**PROPOSITION 4.1.** *Let  $\mathbf{f}$  be an SPP. Let the operator  $\widehat{\mathcal{N}}_{\mathbf{f}}$  be defined as follows:*

$$\widehat{\mathcal{N}}_{\mathbf{f}}(\mathbf{X}) := \mathbf{X} + \sum_{d=0}^{\infty} (\mathbf{f}'(\mathbf{X})^d (\mathbf{f}(\mathbf{X}) - \mathbf{X})) .$$

*Then the sequence  $(\nu^{(k)})_{k \in \mathbb{N}}$  with  $\nu^{(k)} := \widehat{\mathcal{N}}_{\mathbf{f}}^k(\mathbf{0})$  is monotonically increasing, bounded from above by  $\mu \mathbf{f}$  (i.e.  $\nu^{(k)} \leq \mathbf{f}(\nu^{(k)}) \leq \nu^{(k+1)} \leq \mu \mathbf{f}$ ) and converges to  $\mu \mathbf{f}$ .*

Theorem 4.8 below essentially states that  $\widehat{\mathcal{N}}$  can be replaced by  $\mathcal{N}$  without changing the properties of the associated sequence. For the proof of Proposition 4.1 we will need the following generalization of Taylor's theorem.

**LEMMA 4.2.** *Let  $\mathbf{f}$  be an SPP,  $d \in \mathbb{N}$ , and  $\mathbf{0} \leq \mathbf{u}$ , and  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{f}(\mathbf{x})$ . Then*

$$\mathbf{f}^d(\mathbf{x} + \mathbf{u}) \geq \mathbf{f}^d(\mathbf{x}) + \mathbf{f}'(\mathbf{x})^d \mathbf{u} .$$

*In particular, by setting  $\mathbf{u} := \mathbf{f}(\mathbf{x}) - \mathbf{x}$  we get*

$$\mathbf{f}^{d+1}(\mathbf{x}) - \mathbf{f}^d(\mathbf{x}) \geq \mathbf{f}'(\mathbf{x})^d (\mathbf{f}(\mathbf{x}) - \mathbf{x}) .$$

*Proof.* By induction on  $d$ . For  $d = 0$  the statement is trivial. Let  $d \geq 0$ . Then, by Taylor's theorem (Lemma 2.3), we have:

$$\begin{aligned} \mathbf{f}^{d+1}(\mathbf{x} + \mathbf{u}) &= \mathbf{f}(\mathbf{f}^d(\mathbf{x} + \mathbf{u})) \\ &\geq \mathbf{f}(\mathbf{f}^d(\mathbf{x}) + \mathbf{f}'(\mathbf{x})^d \mathbf{u}) && \text{(induction hypothesis)} \\ &\geq \mathbf{f}^{d+1}(\mathbf{x}) + \mathbf{f}'(\mathbf{f}^d(\mathbf{x})) \mathbf{f}'(\mathbf{x})^d \mathbf{u} && \text{(Lemma 2.3)} \\ &\geq \mathbf{f}^{d+1}(\mathbf{x}) + \mathbf{f}'(\mathbf{x})^{d+1} \mathbf{u} && (\mathbf{f}^d(\mathbf{x}) \geq \mathbf{x}) \end{aligned}$$

□

Lemma 4.2 can be used to prove the following.

LEMMA 4.3. *Let  $\mathbf{f}$  be an SPP. Let  $\mathbf{0} \leq \mathbf{x} \leq \mu \mathbf{f}$  and  $\mathbf{x} \leq \mathbf{f}(\mathbf{x})$ . Then*

$$\mathbf{x} + \sum_{d=0}^{\infty} (\mathbf{f}'(\mathbf{x})^d (\mathbf{f}(\mathbf{x}) - \mathbf{x})) \leq \mu \mathbf{f}.$$

*Proof.* Observe that

$$\lim_{d \rightarrow \infty} \mathbf{f}^d(\mathbf{x}) = \mu \mathbf{f} \tag{4.1}$$

because  $\mathbf{0} \leq \mathbf{x} \leq \mu \mathbf{f}$  implies  $\mathbf{f}^d(\mathbf{0}) \leq \mathbf{f}^d(\mathbf{x}) \leq \mu \mathbf{f}$  and as  $(\mathbf{f}^d(\mathbf{0}))_{d \in \mathbb{N}}$  converges to  $\mu \mathbf{f}$  by Theorem 2.4, so does  $(\mathbf{f}^d(\mathbf{x}))_{d \in \mathbb{N}}$ . We have:

$$\begin{aligned} \mathbf{x} + \sum_{d=0}^{\infty} (\mathbf{f}'(\mathbf{x})^d (\mathbf{f}(\mathbf{x}) - \mathbf{x})) &\leq \mathbf{x} + \sum_{d=0}^{\infty} (\mathbf{f}^{d+1}(\mathbf{x}) - \mathbf{f}^d(\mathbf{x})) && \text{(Lemma 4.2)} \\ &= \lim_{d \rightarrow \infty} \mathbf{f}^d(\mathbf{x}) \\ &= \mu \mathbf{f} && \text{(by (4.1))} \end{aligned}$$

□

Now we can prove Proposition 4.1.

*Proof.* [of Proposition 4.1] First we prove the following inequality by induction on  $k$ :

$$\boldsymbol{\nu}^{(k)} \leq \mathbf{f}(\boldsymbol{\nu}^{(k)})$$

The induction base ( $k = 0$ ) is easy. For the step, let  $k \geq 0$ . Then

$$\begin{aligned} \boldsymbol{\nu}^{(k+1)} &= \boldsymbol{\nu}^{(k)} + \sum_{d=0}^{\infty} (\mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)})) \\ &= \mathbf{f}(\boldsymbol{\nu}^{(k)}) + \sum_{d=1}^{\infty} (\mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)})) \\ &= \mathbf{f}(\boldsymbol{\nu}^{(k)}) + \mathbf{f}'(\boldsymbol{\nu}^{(k)}) \sum_{d=0}^{\infty} (\mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)})) \\ &\leq \mathbf{f} \left( \boldsymbol{\nu}^{(k)} + \sum_{d=0}^{\infty} (\mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)})) \right) && \text{(Lemma 2.3)} \\ &= \mathbf{f}(\boldsymbol{\nu}^{(k+1)}). \end{aligned}$$

Now, the inequality  $\boldsymbol{\nu}^{(k)} \leq \mu \mathbf{f}$  follows from Lemma 4.3 by means of a straightforward induction proof. Hence, it follows  $\mathbf{f}(\boldsymbol{\nu}^{(k)}) \leq \mathbf{f}(\mu \mathbf{f}) = \mu \mathbf{f}$ . Further we have

$$\begin{aligned} \mathbf{f}(\boldsymbol{\nu}^{(k)}) &= \boldsymbol{\nu}^{(k)} + (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}) \\ &\leq \boldsymbol{\nu}^{(k)} + \sum_{d=0}^{\infty} \left( \mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}) \right) = \boldsymbol{\nu}^{(k+1)}. \end{aligned} \quad (4.2)$$

So it remains to show that  $(\boldsymbol{\nu}^{(k)})_{k \in \mathbb{N}}$  converges to  $\mu \mathbf{f}$ . As we have already shown  $\boldsymbol{\nu}^{(k)} \leq \mu \mathbf{f}$  it suffices to show that  $\boldsymbol{\kappa}^{(k)} \leq \boldsymbol{\nu}^{(k)}$  because  $(\boldsymbol{\kappa}^{(k)})_{k \in \mathbb{N}}$  converges to  $\mu \mathbf{f}$  by Theorem 2.4. We proceed by induction on  $k$ . The induction base ( $k = 0$ ) is easy. For the step, let  $k \geq 0$ . Then

$$\begin{aligned} \boldsymbol{\kappa}^{(k+1)} &= \mathbf{f}(\boldsymbol{\kappa}^{(k)}) \\ &\leq \mathbf{f}(\boldsymbol{\nu}^{(k)}) && \text{(induction hypothesis)} \\ &\leq \boldsymbol{\nu}^{(k+1)} && \text{(by (4.2))} \end{aligned}$$

This completes the proof.  $\square$

For Theorem 4.8 below we wish to replace the operator  $\widehat{\mathcal{N}}$  in Proposition 4.1 by the Newton operator  $\mathcal{N}$ . To achieve this goal it is convenient to move to the *extended reals*  $\mathbb{R}_{[0, \infty]}$ , i.e., we extend  $\mathbb{R}_{\geq 0}$  by an element  $\infty$  such that addition satisfies  $a + \infty = \infty + a = \infty$  for all  $a \in \mathbb{R}_{\geq 0}$  and multiplication satisfies  $0 \cdot \infty = \infty \cdot 0 = 0$  and  $a \cdot \infty = \infty \cdot a = \infty$  for all  $a \in \mathbb{R}_{\geq 0}$ .

The following proposition will be crucial.

**PROPOSITION 4.4.** *The matrix series  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})^* := \text{Id} + \mathbf{f}'(\boldsymbol{\nu}^{(k)}) + \mathbf{f}'(\boldsymbol{\nu}^{(k)})^2 + \dots$  converges in  $\mathbb{R}_{\geq 0}$  for all Newton approximants  $\boldsymbol{\nu}^{(k)}$ , i.e., there are no  $\infty$  entries.*

In  $\mathbb{R}_{[0, \infty]}$ ,  $\widehat{\mathcal{N}}(\boldsymbol{\nu}^{(k)}) = \boldsymbol{\nu}^{(k)} + \sum_{d=0}^{\infty} (\mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}))$  can be rewritten as  $\boldsymbol{\nu}^{(k)} + \mathbf{f}'(\boldsymbol{\nu}^{(k)})^* (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)})$ . Notice that Proposition 4.4 does not follow trivially from Proposition 4.1, because  $\infty$  entries of  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})^*$  could be cancelled out by matching 0 entries of  $\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}$ .

For the proof of Proposition 4.4 we need several lemmata. In the following, if  $M$  is a matrix we often write  $M_{jk}^i$  resp.  $M_{jk}^*$  when we mean  $(M^i)_{jk}$  resp.  $(M^*)_{jk}$ .

The following lemma assures that a starred matrix has an  $\infty$  entry if and only if it has an  $\infty$  entry on the diagonal.

**LEMMA 4.5.** *Let  $A = (a_{ij}) \in \mathbb{R}_{\geq 0}^{n \times n}$ . Let  $A^*$  have an  $\infty$  entry. Then  $A^*$  also has an  $\infty$  entry on the diagonal, i.e.,  $A_{ii}^* = \infty$  for some  $1 \leq i \leq n$ .*

*Proof.* By induction on  $n$ . The base case  $n = 1$  is clear. For  $n > 1$  assume w.l.o.g. that  $A_{1n}^* = \infty$ . We have

$$A_{1n}^* = A_{11}^* \sum_{j=2}^n a_{1j} (A_{[2..n, 2..n]})_{jn}^*, \quad (4.3)$$

where by  $A_{[2..n, 2..n]}$  we mean the square matrix obtained from  $A$  by erasing the first row and the first column. To see why (4.3) holds, think of  $A_{1n}^*$  as the sum of weights of paths from 1 to  $n$  in the complete graph over the vertices  $\{1, \dots, n\}$ . The weight of a path  $P$  is the product of the weight of  $P$ 's edges, and  $a_{i_1 i_2}$  is the weight of the edge from  $i_1$  to  $i_2$ . Each path  $P$  from 1 to  $n$  can be divided into two sub-paths  $P_1, P_2$  as follows. The second sub-path  $P_2$  is the suffix of  $P$  leading from 1 to  $n$  and not returning to 1. The first sub-path  $P_1$ , possibly empty, is chosen such that  $P = P_1 P_2$ .

Now, the sum of weights of all possible  $P_1$  equals  $A_{11}^*$ , and the sum of weights of all possible  $P_2$  equals  $\sum_{j=2}^n a_{1j}(A_{[2..n,2..n]})_{jn}^*$ . So (4.3) holds.

As  $A_{1n}^* = \infty$ , it follows that either  $A_{11}^*$  or some  $(A_{[2..n,2..n]})_{jn}^*$  equals  $\infty$ . In the first case, we are done. In the second case, by induction, there is an  $i$  such that  $(A_{[2..n,2..n]})_{ii}^* = \infty$ . But then also  $A_{ii}^* = \infty$ , because every entry of  $(A_{[2..n,2..n]})^*$  is less or equal the corresponding entry of  $A^*$ .  $\square$

The following lemma treats the case that  $\mathbf{f}$  is strongly connected (cf. [14]).

LEMMA 4.6. *Let  $\mathbf{f}$  be non-trivially strongly connected. Let  $\mathbf{0} \leq \mathbf{x} \prec \mu\mathbf{f}$ . Then  $\mathbf{f}'(\mathbf{x})^*$  does not have  $\infty$  as an entry.*

*Proof.* By Theorem 2.4 the Kleene sequence  $(\boldsymbol{\kappa}^{(i)})_{i \in \mathbb{N}}$  converges to  $\mu\mathbf{f}$ . Furthermore,  $\boldsymbol{\kappa}^{(i)} \prec \mu\mathbf{f}$  holds for all  $i$ , because, as every component depends non-trivially on itself, any increase in any component results in an increase of the same component in a later Kleene approximant. So, we can choose a Kleene approximant  $\mathbf{y} = \boldsymbol{\kappa}^{(i)}$  such that  $\mathbf{x} \leq \mathbf{y} \prec \mu\mathbf{f}$ . Notice that  $\mathbf{y} \leq \mathbf{f}(\mathbf{y})$ . By monotonicity of  $\mathbf{f}'$  it suffices to show that  $\mathbf{f}'(\mathbf{y})^*$  does not have  $\infty$  as an entry. By Lemma 4.2 (taking  $\mathbf{x} := \mathbf{y}$  and  $\mathbf{u} := \mu\mathbf{f} - \mathbf{y}$ ) we have

$$\mathbf{f}'(\mathbf{y})^d(\mu\mathbf{f} - \mathbf{y}) \leq \mu\mathbf{f} - \mathbf{f}^d(\mathbf{y}).$$

As  $d \rightarrow \infty$ , the right hand side converges to  $\mathbf{0}$ , because, by Kleene's theorem,  $\mathbf{f}^d(\mathbf{y})$  converges to  $\mu\mathbf{f}$ . So the left hand side also converges to  $\mathbf{0}$ . Since  $\mu\mathbf{f} - \mathbf{y} \succ \mathbf{0}$ , every entry of  $\mathbf{f}'(\mathbf{y})^d$  must converge to  $\mathbf{0}$ . Then, by standard facts about matrices (see e.g. [24]), the spectral radius of  $\mathbf{f}'(\mathbf{y})$  is less than 1, i.e.,  $|\lambda| < 1$  for all eigenvalues  $\lambda$  of  $\mathbf{f}'(\mathbf{y})$ . This, in turn, implies that the series  $\mathbf{f}'(\mathbf{y})^* = \text{Id} + \mathbf{f}'(\mathbf{y}) + \mathbf{f}'(\mathbf{y})^2 + \dots$  converges in  $\mathbb{R}_{\geq 0}$ , see [24], page 531. In other words,  $\mathbf{f}'(\mathbf{y})^*$  and hence  $\mathbf{f}'(\mathbf{x})^*$  do not have  $\infty$  as an entry.  $\square$

The following lemma states that Newton's method can only terminate in a component  $s$  after certain other components  $\ell$  have reached  $\mu\mathbf{f}_\ell$ .

LEMMA 4.7. *Let  $1 \leq s, \ell \leq n$ . Let the term  $\mathbf{f}'(\mathbf{X})_{ss}^*$  contain the variable  $X_\ell$ . Let  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{f}(\mathbf{x}) \leq \mu\mathbf{f}$  and  $x_s < \mu\mathbf{f}_s$  and  $x_\ell < \mu\mathbf{f}_\ell$ . Then  $\widehat{N}(\mathbf{x})_s < \mu\mathbf{f}_s$ .*

*Proof.* This proof follows closely a proof of [14]. Let  $d \geq 0$  such that  $\mathbf{f}'(\mathbf{X})_{ss}^d$  contains  $X_\ell$ . Let  $m' \geq 0$  such that  $\mathbf{f}^{m'}(\mathbf{x}) \succ \mathbf{0}$  and  $\mathbf{f}^{m'}(\mathbf{x})_\ell > x_\ell$ . Such an  $m'$  exists because with Kleene's theorem the sequence  $(\mathbf{f}^k(\mathbf{x}))_{k \in \mathbb{N}}$  converges to  $\mu\mathbf{f}$ . Notice that our choice of  $m'$  guarantees  $\mathbf{f}'(\mathbf{f}^{m'}(\mathbf{x}))_{ss}^d > \mathbf{f}'(\mathbf{x})_{ss}^d$ .

Now choose  $m \geq m'$  such that  $\mathbf{f}^{m+1}(\mathbf{x})_s > \mathbf{f}^m(\mathbf{x})_s$ . Such an  $m$  exists because the sequence  $(\mathbf{f}^k(\mathbf{x})_s)_{k \in \mathbb{N}}$  never reaches  $\mu\mathbf{f}_s$ . This is because  $s$  depends on itself (since  $\mathbf{f}'(\mathbf{X})_{ss}^*$  is not constant 0), and so every increase of the  $s$ -component results in an increase of the  $s$ -component in some later iteration of the Kleene sequence.

Now we have

$$\begin{aligned} & \mathbf{f}^{d+m+1}(\mathbf{x}) - \mathbf{f}^{d+m}(\mathbf{x}) \\ & \geq \mathbf{f}'(\mathbf{f}^m(\mathbf{x}))^d(\mathbf{f}^{m+1}(\mathbf{x}) - \mathbf{f}^m(\mathbf{x})) \quad (\text{Lemma 4.2}) \\ & \geq^* \mathbf{f}'(\mathbf{x})^d(\mathbf{f}^{m+1}(\mathbf{x}) - \mathbf{f}^m(\mathbf{x})) \\ & \geq \mathbf{f}'(\mathbf{x})^d \mathbf{f}'(\mathbf{x})^m(\mathbf{f}(\mathbf{x}) - \mathbf{x}) \quad (\text{Lemma 4.2}) \\ & = \mathbf{f}'(\mathbf{x})^{d+m}(\mathbf{f}(\mathbf{x}) - \mathbf{x}). \end{aligned}$$

The inequality marked with  $*$  is strict in the  $s$ -component, due to the choice of  $d$  and  $m$  above. So, with  $b = d + m$  we have:

$$(\mathbf{f}^{b+1}(\mathbf{x}) - \mathbf{f}^b(\mathbf{x}))_s > (\mathbf{f}'(\mathbf{x})^b(\mathbf{f}(\mathbf{x}) - \mathbf{x}))_s \quad (4.4)$$

Again by Lemma 4.2, inequality (4.4) holds for all  $b \in \mathbb{N}$ , but with  $\geq$  instead of  $>$ . Therefore:

$$\begin{aligned} \mu \mathbf{f}_s &= \left( \mathbf{x} + \sum_{i=0}^{\infty} (\mathbf{f}^{i+1}(\mathbf{x}) - \mathbf{f}^i(\mathbf{x})) \right)_s && \text{(Kleene)} \\ &> \left( \mathbf{x} + \mathbf{f}'(\mathbf{x})^* (\mathbf{f}(\mathbf{x}) - \mathbf{x}) \right)_s && \text{(inequality (4.4))} \\ &= \left( \widehat{\mathcal{N}}(\mathbf{x}) \right)_s \end{aligned}$$

□

Now we are ready to prove Proposition 4.4.

*Proof.* [of Proposition 4.4] Using Lemma 4.5 it is enough to show that  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})_{ss}^* \neq \infty$  for all  $s$ . If the  $s$ -component constitutes a trivial SCC then  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})_{ss}^* = 0 \neq \infty$ . So we can assume in the following that the  $s$ -component belongs to a non-trivial SCC, say  $S$ . Let  $\mathbf{X}_L$  be the set of variables contained by the term  $\mathbf{f}'(\mathbf{X})_{ss}^*$ . For any  $t \in S$  we have  $\mathbf{f}'(\mathbf{X})_{ss}^* \geq \mathbf{f}'(\mathbf{X})_{st}^* \mathbf{f}'(\mathbf{X})_{tt}^* \mathbf{f}'(\mathbf{X})_{ts}^*$ . Neither  $\mathbf{f}'(\mathbf{X})_{st}^*$  nor  $\mathbf{f}'(\mathbf{X})_{ts}^*$  is constant zero, because  $S$  is non-trivial. Therefore,  $\mathbf{f}'(\mathbf{X})_{ss}^*$  contains all variables that  $\mathbf{f}'(\mathbf{X})_{tt}^*$  contains, and vice versa, for all  $t \in S$ . So,  $\mathbf{X}_L$  is, for all  $t \in S$ , exactly the set of variables contained by  $\mathbf{f}'(\mathbf{X})_{tt}^*$ .

We distinguish two cases.

**Case 1:** There is a component  $\ell \in L$  such that the sequence  $(\nu_\ell^{(k)})_{k \in \mathbb{N}}$  does not terminate, i.e.,  $\nu_\ell^{(k)} < \mu \mathbf{f}_\ell$  holds for all  $k$ . Then, by Lemma 4.7, the sequence  $(\nu_s^{(k)})_{k \in \mathbb{N}}$  cannot reach  $\mu \mathbf{f}_s$  either. In fact, we have  $\nu_s^{(k)} \prec \mu \mathbf{f}_s$ . Let  $M$  denote the set of those components that the  $S$ -components depend on, but do not depend on  $S$ . In other words,  $M$  contains the components that are “lower” in the DAG of SCCs than  $S$ . Define  $\mathbf{g}(\mathbf{X}_S) := \mathbf{f}_S(\mathbf{X})[M/\mu \mathbf{f}_M]$ . Then  $\mathbf{g}(\mathbf{X}_S)$  is an scSPP with  $\mu \mathbf{g} = \mu \mathbf{f}_S$ . As  $\nu_s^{(k)} \prec \mu \mathbf{g}$ , Lemma 4.6 is applicable, so  $\mathbf{g}'(\nu_s^{(k)})^*$  does not have  $\infty$  as an entry. With  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})_{SS}^* \leq \mathbf{g}'(\nu_s^{(k)})^*$ , we get  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})_{ss}^* < \infty$ , as desired.

**Case 2:** For all components  $\ell \in L$  the sequence  $(\nu_\ell^{(k)})_{k \in \mathbb{N}}$  terminates. Let  $i \in \mathbb{N}$  the least number such that  $\nu_\ell^{(i)} = \mu \mathbf{f}_\ell$  holds for all  $\ell \in L$ . By Lemma 4.7 we have  $\nu_s^{(i)} < \mu \mathbf{f}_s$ . But as, according to Proposition 4.1,  $(\nu_s^{(k)})_{k \in \mathbb{N}}$  converges to  $\mu \mathbf{f}_s$ , there must exist a  $j \geq i$  such that  $0 < (\mathbf{f}'(\boldsymbol{\nu}^{(j)})^* (\mathbf{f}(\boldsymbol{\nu}^{(j)}) - \boldsymbol{\nu}^{(j)}))_s < \infty$ . So there is a component  $u$  with  $0 < \mathbf{f}'(\boldsymbol{\nu}^{(j)})_{su}^* (\mathbf{f}(\boldsymbol{\nu}^{(j)}) - \boldsymbol{\nu}^{(j)})_u < \infty$ . This implies  $0 < \mathbf{f}'(\boldsymbol{\nu}^{(j)})_{su}^* < \infty$ , therefore also  $\mathbf{f}'(\boldsymbol{\nu}^{(j)})_{ss}^* < \infty$ . By monotonicity of  $\mathbf{f}'$ , we have  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})_{ss}^* \leq \mathbf{f}'(\boldsymbol{\nu}^{(j)})_{ss}^* < \infty$  for all  $k \leq j$ . On the other hand, since  $\mathbf{f}'(\mathbf{X})_{ss}^*$  contains only  $L$ -variables and  $\nu_L^{(k)} = \mu \mathbf{f}_L$  holds for all  $k \geq j$ , we also have  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})_{ss}^* = \mathbf{f}'(\boldsymbol{\nu}^{(j)})_{ss}^* < \infty$  for all  $k \geq j$ . □

Now the following theorem follows from Proposition 4.4.

**THEOREM 4.8.** *Let  $\mathbf{f}$  be an SPP. Let the Newton operator  $\mathcal{N}_\mathbf{f}$  be defined as in Definition 3.1:*

$$\mathcal{N}_\mathbf{f}(\mathbf{X}) := \mathbf{X} + (\text{Id} - \mathbf{f}'(\mathbf{X}))^{-1} (\mathbf{f}(\mathbf{X}) - \mathbf{X})$$

1. Then the Newton sequence  $(\boldsymbol{\nu}^{(k)})_{k \in \mathbb{N}}$  with  $\boldsymbol{\nu}^{(k)} = \mathcal{N}_\mathbf{f}^k(\mathbf{0})$  is well-defined (i.e., the matrix inverses exist), monotonically increasing, bounded from above by  $\mu \mathbf{f}$  (i.e.  $\boldsymbol{\nu}^{(k)} \leq \mathbf{f}(\boldsymbol{\nu}^{(k)}) \leq \boldsymbol{\nu}^{(k+1)} \leq \mu \mathbf{f}$ ), and converges to  $\mu \mathbf{f}$ .
2. We have  $(\text{Id} - \mathbf{f}'(\boldsymbol{\nu}^{(k)}))^{-1} = \mathbf{f}'(\boldsymbol{\nu}^{(k)})^*$  for all  $k \in \mathbb{N}$ .  
We also have  $(\text{Id} - \mathbf{f}'(\mathbf{x}))^{-1} = \mathbf{f}'(\mathbf{x})^*$  for all  $\mathbf{x} \prec \mu \mathbf{f}$ .

*Proof.* By Proposition 4.4 the matrix  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})^*$  has no  $\infty$  entries. Then we clearly

have  $\mathbf{f}'(\boldsymbol{\nu}^{(k)})^*(\text{Id} - \mathbf{f}'(\boldsymbol{\nu}^{(k)})) = \text{Id}$ , so  $(\text{Id} - \mathbf{f}'(\boldsymbol{\nu}^{(k)}))^{-1} = \mathbf{f}'(\boldsymbol{\nu}^{(k)})^*$ . Hence, we have

$$\begin{aligned} \widehat{\mathcal{N}}(\boldsymbol{\nu}^{(k)}) &= \boldsymbol{\nu}^{(k)} + \sum_{d=0}^{\infty} \left( \mathbf{f}'(\boldsymbol{\nu}^{(k)})^d (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}) \right) \\ &= \boldsymbol{\nu}^{(k)} + \mathbf{f}'(\boldsymbol{\nu}^{(k)})^* (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}) \\ &= \boldsymbol{\nu}^{(k)} + (\text{Id} - \mathbf{f}'(\boldsymbol{\nu}^{(k)}))^{-1} (\mathbf{f}(\boldsymbol{\nu}^{(k)}) - \boldsymbol{\nu}^{(k)}) \\ &= \mathcal{N}(\boldsymbol{\nu}^{(k)}), \end{aligned}$$

so we can replace  $\widehat{\mathcal{N}}$  by  $\mathcal{N}$ . It remains to show  $(\text{Id} - \mathbf{f}'(\mathbf{x}))^{-1} = \mathbf{f}'(\mathbf{x})^*$  for all  $\mathbf{x} \prec \mu\mathbf{f}$ . It suffices to show that  $\mathbf{f}'(\mathbf{x})^*$  has no  $\infty$  entries. By part 1. the sequence  $(\boldsymbol{\nu}^{(k)})_{k \in \mathbb{N}}$  converges to  $\mu\mathbf{f}$ . So there is a  $k'$  such that  $\mathbf{x} \leq \boldsymbol{\nu}^{(k')}$ . By Proposition 4.4,  $\mathbf{f}'(\boldsymbol{\nu}^{(k')})^*$  has no  $\infty$  entries, so, by monotonicity,  $\mathbf{f}'(\mathbf{x})^*$  has no  $\infty$  entries either.  $\square$

#### 4.2. Monotonicity.

LEMMA 4.9 (Monotonicity of the Newton operator). *Let  $\mathbf{f}$  be an SPP. Let  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y} \leq \mathbf{f}(\mathbf{y}) \leq \mu\mathbf{f}$  and let  $\mathcal{N}_{\mathbf{f}}(\mathbf{y})$  exist. Then*

$$\mathcal{N}_{\mathbf{f}}(\mathbf{x}) \leq \mathcal{N}_{\mathbf{f}}(\mathbf{y}).$$

*Proof.* For  $\mathbf{x} \leq \mathbf{y}$  we have  $\mathbf{f}'(\mathbf{x}) \leq \mathbf{f}'(\mathbf{y})$  as every entry of  $\mathbf{f}'(\mathbf{X})$  is a monotone polynomial. Hence,  $\mathbf{f}'(\mathbf{x})^* \leq \mathbf{f}'(\mathbf{y})^*$ . With this at hand we get:

$$\begin{aligned} \mathcal{N}_{\mathbf{f}}(\mathbf{y}) &= \mathbf{y} + \mathbf{f}'(\mathbf{y})^* (\mathbf{f}(\mathbf{y}) - \mathbf{y}) && \text{(Theorem 4.8)} \\ &\geq \mathbf{y} + \mathbf{f}'(\mathbf{x})^* (\mathbf{f}(\mathbf{y}) - \mathbf{y}) && (\mathbf{f}'(\mathbf{y})^* \geq \mathbf{f}'(\mathbf{x})^*) \\ &\geq \mathbf{y} + \mathbf{f}'(\mathbf{x})^* (\mathbf{f}(\mathbf{x}) + \mathbf{f}'(\mathbf{x})(\mathbf{y} - \mathbf{x}) - \mathbf{y}) && \text{(Lemma 2.3)} \\ &= \mathbf{y} + \mathbf{f}'(\mathbf{x})^* ((\mathbf{f}(\mathbf{x}) - \mathbf{x}) - (\text{Id} - \mathbf{f}'(\mathbf{x}))(\mathbf{y} - \mathbf{x})) \\ &= \mathbf{y} + \mathbf{f}'(\mathbf{x})^* (\mathbf{f}(\mathbf{x}) - \mathbf{x}) - (\mathbf{y} - \mathbf{x}) && (\mathbf{f}'(\mathbf{x})^* = \\ & && (\text{Id} - \mathbf{f}'(\mathbf{x}))^{-1}) \\ &= \mathcal{N}_{\mathbf{f}}(\mathbf{x}) && \text{(Theorem 4.8)} \end{aligned}$$

$\square$

**4.3. Exponential Convergence Order in the Nonsingular Case.** If the matrix  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is nonsingular, Newton's method has exponential convergence order in the sense of Definition 2.8.<sup>3</sup> This is, in fact, a well known general property of Newton's method, see e.g. [27]. For completeness, we show that Newton's method for "nonsingular" SPPs has exponential convergence order, see Theorem 4.13 below.

LEMMA 4.10. *Let  $\mathbf{f}$  be an SPP. Let  $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{f}$  such that  $\mathbf{f}'(\mathbf{x})^*$  exists. Then there is a bilinear function  $B : \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$  with*

$$\mu\mathbf{f} - \mathcal{N}(\mathbf{x}) \leq \mathbf{f}'(\mathbf{x})^* B(\mu\mathbf{f} - \mathbf{x}, \mu\mathbf{f} - \mathbf{x}).$$

<sup>3</sup>In numerical analysis, the terms "quadratic convergence" or "Q-quadratic convergence" are commonly used, see e.g. [27]. It means that the error  $e'$  of the new approximant is bounded by  $c \cdot e^2$  where  $e$  is the error of the old approximant and  $c > 0$  is some constant. "Quadratic convergence" implies exponential convergence order in the sense of Definition 2.8, and we avoid the notion of "quadratic convergence" in the following.

*Proof.* Write  $\mathbf{d} := \mu\mathbf{f} - \mathbf{x}$ . By Taylor's theorem (cf. Lemma 2.3) we obtain

$$\mathbf{f}(\mathbf{x} + \mathbf{d}) \leq \mathbf{f}(\mathbf{x}) + \mathbf{f}'(\mathbf{x})\mathbf{d} + B(\mathbf{d}, \mathbf{d}) \quad (4.5)$$

for the bilinear map  $B(\mathbf{X}) := \mathbf{f}''(\mu\mathbf{f})(\mathbf{X}, \mathbf{X})$ , where  $\mathbf{f}''(\mu\mathbf{f})$  denotes the rank-3 tensor of the second partial derivatives evaluated at  $\mu\mathbf{f}$  [27]. We have

$$\begin{aligned} \mu\mathbf{f} - \mathcal{N}(\mathbf{x}) &= \mathbf{d} - \mathbf{f}'(\mathbf{x})^*(\mathbf{f}(\mathbf{x}) - \mathbf{x}) \\ &= \mathbf{d} - \mathbf{f}'(\mathbf{x})^*(\mathbf{d} + \mathbf{f}(\mathbf{x}) - (\mathbf{x} + \mathbf{d})) \\ &= \mathbf{d} - \mathbf{f}'(\mathbf{x})^*(\mathbf{d} + \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x} + \mathbf{d})) \quad (\mathbf{x} + \mathbf{d} = \mu\mathbf{f} = \mathbf{f}(\mu\mathbf{f})) \\ &\leq \mathbf{d} - \mathbf{f}'(\mathbf{x})^*(\mathbf{d} - \mathbf{f}'(\mathbf{x})\mathbf{d} - B(\mathbf{d}, \mathbf{d})) \quad (\text{by (4.5)}) \\ &= \mathbf{d} - \mathbf{f}'(\mathbf{x})^*((\text{Id} - \mathbf{f}'(\mathbf{x}))\mathbf{d} - B(\mathbf{d}, \mathbf{d})) \\ &= \mathbf{d} - \mathbf{d} + \mathbf{f}'(\mathbf{x})^*B(\mathbf{d}, \mathbf{d}) \quad (\mathbf{f}'(\mathbf{x})^* = (\text{Id} - \mathbf{f}'(\mathbf{x}))^{-1}) \\ &= \mathbf{f}'(\mathbf{x})^*B(\mathbf{d}, \mathbf{d}) \end{aligned}$$

□

Define for the following lemmata  $\Delta^{(k)} := \mu\mathbf{f} - \nu^{(k)}$ , i.e.,  $\Delta^{(k)}$  is the error after  $k$  Newton iterations. The following lemma bounds  $\|\Delta^{(k+1)}\|$  in terms of  $\|\Delta^{(k)}\|^2$  if  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is nonsingular.

LEMMA 4.11. *Let  $\mathbf{f}$  be an SPP such that  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is nonsingular. Then there is a constant  $c > 0$  such that*

$$\|\Delta^{(k+1)}\| \leq c \cdot \|\Delta^{(k)}\|^2 \quad \text{for all } k \in \mathbb{N}.$$

*Proof.* As  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is nonsingular, we have, by Theorem 4.8,  $(\text{Id} - \mathbf{f}'(\mathbf{x}))^{-1} = \mathbf{f}'(\mathbf{x})^*$  for all  $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{f}$ . By continuity, there is a  $c_1 > 0$  such that  $\|\mathbf{f}'(\mathbf{x})^*\| \leq c_1$  for all  $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{f}$ . Similarly, there is a  $c_2 > 0$  such that  $\|B(\mathbf{x}, \mathbf{x})\| \leq c_2 \|\mathbf{x}\|^2$  for all  $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{f}$ , because  $B$  is bilinear. So it follows from Lemma 4.10 that  $\|\Delta^{(k+1)}\| \leq c_1 c_2 \|\Delta^{(k)}\|^2$ . □

Lemma 4.11 can be used to show that the error  $\Delta^{(i)}$  decays double-exponentially in the nonsingular case:

LEMMA 4.12. *Let  $\mathbf{f}$  be an SPP such that  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is nonsingular. Then there is a constant  $\tilde{k}_{\mathbf{f}} \in \mathbb{N}$  such that for all  $i \in \mathbb{N}$*

$$\|\Delta^{(\tilde{k}_{\mathbf{f}}+i)}\| \leq 2^{-2^i} \quad \text{for all } i \in \mathbb{N}.$$

*Proof.* We can assume w.l.o.g. that  $c \geq 1$  for the  $c$  from Lemma 4.11. As the  $\Delta^{(k)}$  converge to  $\mathbf{0}$ , we can choose  $\tilde{k}_{\mathbf{f}} \in \mathbb{N}$  large enough such that  $d := -\log \|\Delta^{(\tilde{k}_{\mathbf{f}})}\| - \log c \geq 1$ . As  $c, d \geq 1$ , it suffices to show the following inequality:

$$\|\Delta^{(\tilde{k}_{\mathbf{f}}+i)}\| \leq \frac{2^{-d \cdot 2^i}}{c}.$$

We proceed by induction on  $i$ . For  $i = 0$ , the inequality above follows from the definition of  $d$ . Let  $i \geq 0$ . Then

$$\|\Delta^{(\tilde{k}_{\mathbf{f}}+i+1)}\| \leq c \cdot \|\Delta^{(\tilde{k}_{\mathbf{f}}+i)}\|^2 \quad (\text{Lemma 4.11})$$

$$\begin{aligned}
&\leq c \cdot \frac{2^{-d \cdot 2^i \cdot 2}}{c^2} && \text{(induction hypothesis)} \\
&= \frac{2^{-d \cdot 2^{i+1}}}{c}.
\end{aligned}$$

□

Now it follows easily that Newton's method has an exponential convergence order in the nonsingular case. More precisely:

**THEOREM 4.13.** *Let  $\mathbf{f}$  be an SPP such that  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is nonsingular. Then there is a constant  $k_{\mathbf{f}} \in \mathbb{N}$  such that*

$$\beta(k_{\mathbf{f}} + i) \geq 2^i \text{ for all } i \in \mathbb{N}.$$

*Proof.* Choose  $m \in \mathbb{N}$  large enough such that  $2^{m+i} + \log(\mu\mathbf{f}_j) \geq 2^i$  holds for all components  $j$ . Thus

$$\begin{aligned}
\Delta_j^{(\tilde{k}_{\mathbf{f}}+m+i)} / \mu\mathbf{f}_j &\leq 2^{-2^{m+i}} / \mu\mathbf{f}_j && \text{(Lemma 4.12 with } \|\cdot\|_{\infty}\text{-norm)} \\
&= 2^{-(2^{m+i} + \log(\mu\mathbf{f}_j))} \\
&\leq 2^{-2^i} && \text{(choice of } m\text{)}.
\end{aligned}$$

So, with  $k_{\mathbf{f}} := \tilde{k}_{\mathbf{f}} + m$ , the approximant  $\nu^{(k_{\mathbf{f}}+i)}$  has at least  $2^i$  valid bits of  $\mu\mathbf{f}$ . □

This type of analysis has serious shortcomings. In particular, Theorem 4.13 excludes the case where  $\text{Id} - \mathbf{f}'(\mu\mathbf{f})$  is singular. We will include this case in our convergence analysis in § 5 and § 6. Furthermore, and maybe more severely, Theorem 4.13 does not give any bound on  $k_{\mathbf{f}}$ . We solve this problem for strongly connected SPPs in § 5.

**4.4. Reduction to the Quadratic Case.** In this section we reduce SPPs to quadratic SPPs, i.e., to SPPs in which every polynomial  $f_i(\mathbf{X})$  has degree at most 2, and show that the convergence on the quadratic SPP is no faster than on the original SPP. In the following sections we will obtain convergence speed guarantees of Newton's method on quadratic SPPs. Hence, one can perform Newton's method on the original SPP and, using the results of this section, convergence is at least as fast as on the corresponding quadratic SPP.

The idea to reduce the degree of our SPP  $\mathbf{f}$  is to introduce auxiliary variables that express quadratic subterms. This can be done repeatedly until all polynomials in the system have reached degree at most 2. The construction is very similar to the one that transforms a context-free grammar into another grammar in Chomsky normal form. The following theorem shows that the transformation does not accelerate the convergence of Newton's method.

**THEOREM 4.14.** *Let  $\mathbf{f}(\mathbf{X})$  be an SPP such that  $f_s(\mathbf{X}) = g(\mathbf{X}) + h(\mathbf{X})X_iX_j$  for some  $1 \leq i, j, s \leq n$ , where  $g(\mathbf{X})$  and  $h(\mathbf{X})$  are polynomials with nonnegative coefficients. Let  $\tilde{\mathbf{f}}(\mathbf{X}, Y)$  be the SPP given by*

$$\begin{aligned}
\tilde{f}_{\ell}(\mathbf{X}, Y) &= f_{\ell}(\mathbf{X}) && \text{for every } \ell \in \{1, \dots, s-1\} \\
\tilde{f}_s(\mathbf{X}, Y) &= g(\mathbf{X}) + h(\mathbf{X})Y \\
\tilde{f}_{\ell}(\mathbf{X}, Y) &= f_{\ell}(\mathbf{X}) && \text{for every } \ell \in \{s+1, \dots, n\} \\
\tilde{f}_{n+1}(\mathbf{X}, Y) &= X_iX_j.
\end{aligned}$$



Then the function  $b : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$  given by  $b(\mathbf{X}) = (X_1, \dots, X_n, X_i X_j)^\top$  is a bijection between the set of fixed points of  $\mathbf{f}(\mathbf{X})$  and  $\tilde{\mathbf{f}}(\mathbf{X}, Y)$ . Moreover,  $\tilde{\mathbf{v}}^{(k)} \leq (\nu_1^{(k)}, \dots, \nu_n^{(k)}, \nu_i^{(k)} \nu_j^{(k)})^\top$  for all  $k \in \mathbb{N}$ , where  $\tilde{\mathbf{v}}^{(k)}$  and  $\nu^{(k)}$  are the Newton approximants of  $\tilde{\mathbf{f}}$  and  $\mathbf{f}$ , respectively.

*Proof.* We first show the claim regarding  $b$ : if  $\mathbf{x}$  is a fixed point of  $\tilde{\mathbf{f}}$ , then  $b(\mathbf{x}) = (\mathbf{x}, x_i x_j)$  is a fixed point of  $\mathbf{f}$ . Conversely, if  $(\mathbf{x}, y)$  is a fixed point of  $\mathbf{f}$ , then we have  $y = x_i x_j$  implying that  $\mathbf{x}$  is a fixed point of  $\tilde{\mathbf{f}}$ . Therefore, the least fixed point  $\mu \mathbf{f}$  of  $\mathbf{f}$  determines  $\mu \tilde{\mathbf{f}}$ , and vice versa.

Now we show that the Newton sequence of  $\mathbf{f}$  converges at least as fast as the Newton sequence of  $\tilde{\mathbf{f}}$ . In the following we write  $\mathbf{Y}$  for the  $(n+1)$ -dimensional vector of variables  $(X_1, \dots, X_n, Y)^\top$  and, as usual,  $\mathbf{X}$  for  $(X_1, \dots, X_n)^\top$ . For an  $(n+1)$ -dimensional vector  $\mathbf{x}$ , we let  $\mathbf{x}_{[1,n]}$  denote its restriction to the  $n$  first components, i.e.,  $\mathbf{x}_{[1,n]} := (x_1, \dots, x_n)^\top$ . Note that  $\mathbf{Y}_{[1,n]} = \mathbf{X}$ . Let  $\mathbf{e}_s$  denote the unit vector  $(0, \dots, 0, 1, 0, \dots, 0)^\top$ , where the “1” is on the  $s$ -th place. We have:

$$\tilde{\mathbf{f}}(\mathbf{Y}) = \begin{pmatrix} \mathbf{f}(\mathbf{X}) + \mathbf{e}_s h(\mathbf{X})(Y - X_i X_j) \\ X_i X_j \end{pmatrix}$$

and

$$\tilde{\mathbf{f}}'(\mathbf{Y}) = \begin{pmatrix} \mathbf{f}'(\mathbf{X}) + \mathbf{e}_s \partial_{\mathbf{X}} h(\mathbf{X})(Y - X_i X_j) & \mathbf{e}_s h(\mathbf{X}) \\ \partial_{\mathbf{X}} X_i X_j & 0 \end{pmatrix}$$

We need the following lemma.

LEMMA 4.15. Let  $\mathbf{z} \in \mathbb{R}_{\geq 0}^n$ ,  $\boldsymbol{\delta} = (\text{Id} - \mathbf{f}'(\mathbf{z}))^{-1}(\mathbf{f}(\mathbf{z}) - \mathbf{z})$  and  $\tilde{\boldsymbol{\delta}} = (\text{Id} - \tilde{\mathbf{f}}'(\mathbf{z}, z_i z_j))^{-1}(\tilde{\mathbf{f}}(\mathbf{z}, z_i z_j) - (\mathbf{z}, z_i z_j)^\top)$ . Then  $\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}_{[1,n]}$ .

*Proof of the lemma.*

$$\begin{aligned} \tilde{\mathbf{f}}'(\mathbf{z}, z_i z_j) &= \begin{pmatrix} \mathbf{f}'(\mathbf{z}) + \mathbf{e}_s h(\mathbf{z}) \partial_{\mathbf{X}}(Y - X_i X_j)|_{\mathbf{Y}=(\mathbf{z}, z_i z_j)} & \mathbf{e}_s h(\mathbf{z}) \\ \partial_{\mathbf{X}} X_i X_j|_{\mathbf{Y}=(\mathbf{z}, z_i z_j)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{f}'(\mathbf{z}) - \mathbf{e}_s h(\mathbf{z}) \partial_{\mathbf{X}}(X_i X_j)|_{\mathbf{X}=\mathbf{z}} & \mathbf{e}_s h(\mathbf{z}) \\ \partial_{\mathbf{X}} X_i X_j|_{\mathbf{X}=\mathbf{z}} & 0 \end{pmatrix} \end{aligned}$$

We have  $(\text{Id} - \tilde{\mathbf{f}}'(\mathbf{z}, z_i z_j))\tilde{\boldsymbol{\delta}} = (\tilde{\mathbf{f}}(\mathbf{z}, z_i z_j) - (\mathbf{z}, z_i z_j)^\top)$ , or equivalently:

$$\begin{pmatrix} \text{Id} - \mathbf{f}'(\mathbf{z}) + \mathbf{e}_s h(\mathbf{z}) \partial_{\mathbf{X}}(X_i X_j)|_{\mathbf{X}=\mathbf{z}} & -\mathbf{e}_s h(\mathbf{z}) \\ -\partial_{\mathbf{X}} X_i X_j|_{\mathbf{X}=\mathbf{z}} & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\boldsymbol{\delta}}_{[1,n]} \\ \tilde{\delta}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{z}) - \mathbf{z} \\ 0 \end{pmatrix}$$

Multiplying the last row by  $\mathbf{e}_s h(\mathbf{z})$  and adding to the first  $n$  rows yields:

$$(\text{Id} - \mathbf{f}'(\mathbf{z}))\tilde{\boldsymbol{\delta}}_{[1,n]} = \mathbf{f}(\mathbf{z}) - \mathbf{z}$$

So we have  $\tilde{\boldsymbol{\delta}}_{[1,n]} = (\text{Id} - \mathbf{f}'(\mathbf{z}))^{-1}(\mathbf{f}(\mathbf{z}) - \mathbf{z}) = \boldsymbol{\delta}$ , which proves the lemma.

Now we proceed by induction on  $k$  to show  $\tilde{\mathbf{v}}_{[1,n]}^{(k)} \leq \nu^{(k)}$ , where  $\tilde{\mathbf{v}}^{(k)}$  is the Newton sequence for  $\tilde{\mathbf{f}}$ . By definition of the Newton sequence this is true for  $k = 0$ . For the step, let  $k \geq 0$  and define  $\mathbf{u} := (\tilde{\mathbf{v}}_{[1,n]}^{(k)}, \tilde{\nu}_i^{(k)} \cdot \tilde{\nu}_j^{(k)})^\top$ . Then we have:

$$\tilde{\mathbf{v}}_{[1,n]}^{(k+1)} = \mathcal{N}_{\tilde{\mathbf{f}}}(\tilde{\mathbf{v}}^{(k)})_{[1,n]}$$

$$\begin{aligned}
& \stackrel{(*)}{\leq} \mathcal{N}_{\tilde{\mathbf{f}}}(\mathbf{u})_{[1,n]} && \text{(see below)} \\
& = \tilde{\mathbf{v}}_{[1,n]}^{(k)} + \left( (\text{Id} - \tilde{\mathbf{f}}'(\mathbf{u}))^{-1} (\tilde{\mathbf{f}}(\mathbf{u}) - \mathbf{u}) \right)_{[1,n]} \\
& = \tilde{\mathbf{v}}_{[1,n]}^{(k)} + (\text{Id} - \mathbf{f}'(\tilde{\mathbf{v}}_{[1,n]}^{(k)}))^{-1} (\mathbf{f}(\tilde{\mathbf{v}}_{[1,n]}^{(k)}) - \tilde{\mathbf{v}}_{[1,n]}^{(k)}) && \text{(Lemma 4.15)} \\
& = \mathcal{N}_{\mathbf{f}}(\tilde{\mathbf{v}}_{[1,n]}^{(k)}) \\
& \leq \mathcal{N}_{\mathbf{f}}(\boldsymbol{\nu}^{(k)}) && \text{(induction)} \\
& = \boldsymbol{\nu}^{(k+1)}
\end{aligned}$$

At the inequality marked with  $(*)$  we used the monotonicity of  $\mathcal{N}_{\tilde{\mathbf{f}}}$  (Lemma 4.9) combined with Theorem 4.8, which states  $\tilde{\mathbf{v}}^{(k)} \leq \tilde{\mathbf{f}}(\tilde{\mathbf{v}}^{(k)})$ , hence in particular  $\tilde{\nu}_{n+1}^{(k)} \leq \tilde{\nu}_i^{(k)} \tilde{\nu}_j^{(k)}$ . This concludes the proof of Theorem 4.14.  $\square$

**5. Strongly Connected SPPs.** In this section we study the convergence speed of Newton's method on strongly connected SPPs, short scSPPs, see Definition 2.7.

**5.1. Cone Vectors.** Our convergence speed analysis makes crucial use of the existence of *cone vectors*.

DEFINITION 5.1. *Let  $\mathbf{f}$  be an SPP. A vector  $\mathbf{d} \in \mathbb{R}_{\geq 0}^n$  is a cone vector if  $\mathbf{d} \succ \mathbf{0}$  and  $\mathbf{f}'(\mu\mathbf{f})\mathbf{d} \leq \mathbf{d}$ .*

We will show that any scSPP has a cone vector, see Proposition 5.4 below. As a first step, we show the following lemma.

LEMMA 5.2. *Any scSPP  $\mathbf{f}$  has a vector  $\mathbf{d} > \mathbf{0}$  with  $\mathbf{f}'(\mu\mathbf{f})\mathbf{d} \leq \mathbf{d}$ .*

*Proof.* Consider the Kleene sequence  $(\boldsymbol{\kappa}^{(k)})_{k \in \mathbb{N}}$ . We have  $\mathbf{0} \leq \boldsymbol{\kappa}^{(k)} \prec \mu\mathbf{f}$  for all  $k \in \mathbb{N}$ . By Theorem 4.8.2., the matrices  $(\text{Id} - \mathbf{f}'(\boldsymbol{\kappa}^{(k)}))^{-1} = \mathbf{f}'(\boldsymbol{\kappa}^{(k)})^*$  exist for all  $k$ . Let  $\|\cdot\|$  be any norm. Define the vectors

$$\mathbf{d}^{(k)} := \frac{\mathbf{f}'(\boldsymbol{\kappa}^{(k)})^* \mathbf{1}}{\|\mathbf{f}'(\boldsymbol{\kappa}^{(k)})^* \mathbf{1}\|}.$$

Notice that for all  $k \in \mathbb{N}$  we have  $(\text{Id} - \mathbf{f}'(\boldsymbol{\kappa}^{(k)}))\mathbf{d}^{(k)} = \frac{\mathbf{1}}{\|\mathbf{f}'(\boldsymbol{\kappa}^{(k)})^* \mathbf{1}\|} \geq \mathbf{0}$ . Furthermore we have  $\mathbf{d}^{(k)} \in C$ , where  $C := \{\mathbf{x} \geq \mathbf{0} \mid \|\mathbf{x}\| = 1\}$  is compact. So the sequence  $(\mathbf{d}^{(k)})_{k \in \mathbb{N}}$  has a convergent subsequence, whose limit, say  $\mathbf{d}$ , is also in  $C$ . In particular  $\mathbf{d} > \mathbf{0}$ . As  $(\boldsymbol{\kappa}^{(k)})_{k \in \mathbb{N}}$  converges to  $\mu\mathbf{f}$  and  $(\text{Id} - \mathbf{f}'(\boldsymbol{\kappa}^{(k)}))\mathbf{d}^{(k)} \geq \mathbf{0}$ , it follows by continuity  $(\text{Id} - \mathbf{f}'(\mu\mathbf{f}))\mathbf{d} \geq \mathbf{0}$ .  $\square$

LEMMA 5.3. *Let  $\mathbf{f}$  be an scSPP and let  $\mathbf{d} > \mathbf{0}$  with  $\mathbf{f}'(\mu\mathbf{f})\mathbf{d} \leq \mathbf{d}$ . Then  $\mathbf{d}$  is a cone vector, i.e.,  $\mathbf{d} \succ \mathbf{0}$ .*

*Proof.* Since  $\mathbf{f}$  is an SPP, every component of  $\mathbf{f}'(\mu\mathbf{f})$  is nonnegative. So,

$$\mathbf{0} \leq \mathbf{f}'(\mu\mathbf{f})^n \mathbf{d} \leq \mathbf{f}'(\mu\mathbf{f})^{n-1} \mathbf{d} \leq \dots \leq \mathbf{f}'(\mu\mathbf{f}) \mathbf{d} \leq \mathbf{d}.$$

Let w.l.o.g.  $d_1 > 0$ . As  $\mathbf{f}$  is strongly connected, there is for all  $j$  with  $1 \leq j \leq n$  an  $r_j \leq n$  such that  $(\mathbf{f}'(\mu\mathbf{f})^{r_j})_{j1} > 0$ . Hence,  $(\mathbf{f}'(\mu\mathbf{f})^{r_j} \mathbf{d})_j > 0$  for all  $j$ . With above inequality chain, it follows that  $d_j \geq (\mathbf{f}'(\mu\mathbf{f})^{r_j} \mathbf{d})_j > 0$ . So,  $\mathbf{d} \succ \mathbf{0}$ .  $\square$

The following proposition follows immediately by combining Lemmata 5.2 and 5.3.

PROPOSITION 5.4. *Any scSPP has a cone vector.*

We remark that using Perron-Frobenius theory [3] there is a simpler proof for Proposition 5.4: By Theorem 4.8  $\mathbf{f}'(\mathbf{x})^*$  exists for all  $\mathbf{x} \prec \mathbf{f}$ . So, by fundamental

matrix facts [3], the spectral radius of  $\mathbf{f}'(\mathbf{x})$  is less than 1 for all  $\mathbf{x} \prec \mu\mathbf{f}$ . As the eigenvalues of a matrix depend continuously on the matrix, the spectral radius of  $\mathbf{f}'(\mu\mathbf{f})$ , say  $\rho$ , is at most 1. Since  $\mathbf{f}$  is strongly connected,  $\mathbf{f}'(\mu\mathbf{f})$  is irreducible, and so Perron-Frobenius theory guarantees the existence of an eigenvector  $\mathbf{d} \succ \mathbf{0}$  of  $\mathbf{f}'(\mu\mathbf{f})$  with eigenvalue  $\rho$ . So we have  $\mathbf{f}'(\mu\mathbf{f})\mathbf{d} = \rho\mathbf{d} \leq \mathbf{d}$ , i.e., the eigenvector  $\mathbf{d}$  is a cone vector.

**5.2. Convergence Speed in Terms of Cone Vectors.** Now we show that cone vectors play a fundamental role for the convergence speed of Newton's method. The following lemma gives a lower bound of the Newton approximant  $\nu^{(1)}$  in terms of a cone vector.

LEMMA 5.5. *Let  $\mathbf{f}$  be a (not necessarily clean) SPP such that  $\mathbf{f}'(\mathbf{0})^*$  exists. Let  $\mathbf{d}$  be a cone vector of  $\mathbf{f}$ . Let  $\mathbf{0} \geq \mu\mathbf{f} - \lambda\mathbf{d}$  for some  $\lambda \geq 0$ . Then*

$$\mathcal{N}(\mathbf{0}) \geq \mu\mathbf{f} - \frac{1}{2}\lambda\mathbf{d}.$$

*Proof.* We write  $\mathbf{f}(\mathbf{X})$  as a sum

$$\mathbf{f}(\mathbf{X}) = \mathbf{c} + \sum_{k=1}^D L_k(\mathbf{X}, \dots, \mathbf{X})\mathbf{X}$$

where  $D$  is the degree of  $\mathbf{f}$ , and every  $L_k$  is a  $(k-1)$ -linear map from  $(\mathbb{R}^n)^{k-1}$  to  $\mathbb{R}^{n \times n}$ . Notice that  $\mathbf{f}'(\mathbf{X}) = \sum_{k=1}^D k \cdot L_k(\mathbf{X}, \dots, \mathbf{X})$ . We write  $L$  for  $L_1$ , and  $\mathbf{h}(\mathbf{X})$  for  $\mathbf{f}(\mathbf{X}) - L\mathbf{X} - \mathbf{c}$ .

$$\begin{aligned} \frac{\lambda}{2}\mathbf{d} &= \frac{\lambda}{2}(L^*\mathbf{d} - L^*L\mathbf{d}) && (L^* = \text{Id} + L^*L) \\ &\geq \frac{\lambda}{2}(L^*\mathbf{f}'(\mu\mathbf{f})\mathbf{d} - L^*L\mathbf{d}) && (\mathbf{f}'(\mu\mathbf{f})\mathbf{d} \leq \mathbf{d}) \\ &= \frac{\lambda}{2}L^*\mathbf{h}'(\mu\mathbf{f})\mathbf{d} && (\mathbf{f}'(\mathbf{x}) = \mathbf{h}'(\mathbf{x}) + L) \\ &= L^*\frac{1}{2}\mathbf{h}'(\mu\mathbf{f})\lambda\mathbf{d} \\ &\geq L^*\frac{1}{2}\mathbf{h}'(\mu\mathbf{f})\mu\mathbf{f} && (\lambda\mathbf{d} \geq \mu\mathbf{f}) \\ &= L^*\frac{1}{2}\sum_{k=2}^D k \cdot L_k(\mu\mathbf{f}, \dots, \mu\mathbf{f})\mu\mathbf{f} \\ &\geq L^*\sum_{k=2}^D L_k(\mu\mathbf{f}, \dots, \mu\mathbf{f})\mu\mathbf{f} \\ &= L^*\mathbf{h}(\mu\mathbf{f}) \\ &= L^*(\mathbf{f}(\mu\mathbf{f}) - L\mu\mathbf{f} - \mathbf{c}) && (\mathbf{f}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) + L\mathbf{x} + \mathbf{c}) \\ &= L^*\mu\mathbf{f} - L^*L\mu\mathbf{f} - L^*\mathbf{c} && (\mathbf{f}(\mu\mathbf{f}) = \mu\mathbf{f}) \\ &= \mu\mathbf{f} - L^*\mathbf{c} && (L^* = \text{Id} + L^*L) \\ &= \mu\mathbf{f} - \mathcal{N}(\mathbf{0}) && (\mathcal{N}(\mathbf{0}) = \mathbf{f}'(\mathbf{0})^*\mathbf{f}(\mathbf{0}) = L^*\mathbf{c}) \end{aligned}$$

□

We extend Lemma 5.5 to arbitrary vectors  $\mathbf{x}$  as follows.

LEMMA 5.6. *Let  $\mathbf{f}$  be a (not necessarily clean) SPP. Let  $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{f}$  and  $\mathbf{x} \leq \mathbf{f}(\mathbf{x})$  such that  $\mathbf{f}'(\mathbf{x})^*$  exists. Let  $\mathbf{d}$  be a cone vector of  $\mathbf{f}$ . Let  $\mathbf{x} \geq \mu\mathbf{f} - \lambda\mathbf{d}$  for some  $\lambda \geq 0$ . Then*

$$\mathcal{N}(\mathbf{x}) \geq \mu\mathbf{f} - \frac{1}{2}\lambda\mathbf{d}.$$

*Proof.* Define  $\mathbf{g}(\mathbf{X}) := \mathbf{f}(\mathbf{X} + \mathbf{x}) - \mathbf{x}$ . We first show that  $\mathbf{g}$  is an SPP (not necessarily clean). The only coefficients of  $\mathbf{g}$  that could be negative are those of degree 0. But we have  $\mathbf{g}(\mathbf{0}) = \mathbf{f}(\mathbf{x}) - \mathbf{x} \geq \mathbf{0}$ , and so these coefficients are also nonnegative.

It follows immediately from the definition that  $\mu\mathbf{f} - \mathbf{x} \geq \mathbf{0}$  is the least fixed point of  $\mathbf{g}$ . Moreover,  $\mathbf{g}$  satisfies  $\mathbf{g}'(\mu\mathbf{f} - \mathbf{x})\mathbf{d} \leq \mathbf{d}$ , and so  $\mathbf{d}$  is also a cone vector of  $\mathbf{g}$ . Finally, we have  $\mathbf{0} \geq \mu\mathbf{f} - \mathbf{x} - \lambda\mathbf{d} = \mu\mathbf{g} - \lambda\mathbf{d}$ . So, Lemma 5.5 can be applied as follows.

$$\begin{aligned} \mathcal{N}_{\mathbf{f}}(\mathbf{x}) &= \mathbf{x} + \mathbf{f}'(\mathbf{x})^*(\mathbf{f}(\mathbf{x}) - \mathbf{x}) \\ &= \mathbf{x} + \mathbf{g}'(\mathbf{0})^*(\mathbf{g}(\mathbf{0}) - \mathbf{0}) \\ &= \mathbf{x} + \mathcal{N}_{\mathbf{g}}(\mathbf{0}) \\ &\geq \mathbf{x} + \mu\mathbf{g} - \frac{1}{2}\lambda\mathbf{d} && \text{(Lemma 5.5)} \\ &= \mu\mathbf{f} - \frac{1}{2}\lambda\mathbf{d} \end{aligned}$$

□

By induction we can extend this lemma to the whole Newton sequence:

LEMMA 5.7. *Let  $\mathbf{d}$  be a cone vector of an SPP  $\mathbf{f}$  and let  $\lambda_{max} = \max_j \{\frac{\mu\mathbf{f}_j}{d_j}\}$ . Then*

$$\boldsymbol{\nu}^{(k)} \geq \mu\mathbf{f} - 2^{-k}\lambda_{max}\mathbf{d}.$$

Before proving the lemma we illustrate it by a picture. The dashed line in Figure 5.1 is the ray  $\mathbf{r}(t) = \mu\mathbf{f} - t\mathbf{d}$  along a cone vector  $\mathbf{d}$ . Notice that  $\mathbf{r}(0)$  equals  $\mu\mathbf{f}$  and  $\mathbf{r}(\lambda_{max})$  is the greatest point on the ray that is below  $\mathbf{0}$ . The figure also shows the Newton iterates  $\boldsymbol{\nu}^{(k)}$  for  $0 \leq k \leq 2$  (shape:  $\times$ ) and the corresponding points  $\mathbf{r}(2^{-k}\lambda_{max})$  (shape:  $+$ ) located on the ray  $\mathbf{r}$ . Observe that  $\boldsymbol{\nu}^{(k)} \geq \mathbf{r}(2^{-k}\lambda_{max})$ , as claimed by Lemma 5.7.

*Proof.* [of Lemma 5.7] By induction on  $k$ . For the induction base ( $k = 0$ ) we have for all components  $i$ :

$$(\mu\mathbf{f} - \lambda_{max}\mathbf{d})_i = \left( \mu\mathbf{f} - \max_j \left\{ \frac{\mu\mathbf{f}_j}{d_j} \right\} \mathbf{d} \right)_i \leq \mu\mathbf{f}_i - \frac{\mu\mathbf{f}_i}{d_i}d_i = 0,$$

so  $\boldsymbol{\nu}^{(0)} = \mathbf{0} \geq \mu\mathbf{f} - \lambda_{max}\mathbf{d}$ .

For the induction step, let  $k \geq 0$ . By induction hypothesis we have  $\boldsymbol{\nu}^{(k)} \geq \mu\mathbf{f} - 2^{-k}\lambda_{max}\mathbf{d}$ . So we can apply Lemma 5.6 to get

$$\boldsymbol{\nu}^{(k+1)} = \mathcal{N}(\boldsymbol{\nu}^{(k)}) \geq \mu\mathbf{f} - \frac{1}{2}2^{-k}\lambda_{max}\mathbf{d} = \mu\mathbf{f} - 2^{-(k+1)}\lambda_{max}\mathbf{d}.$$

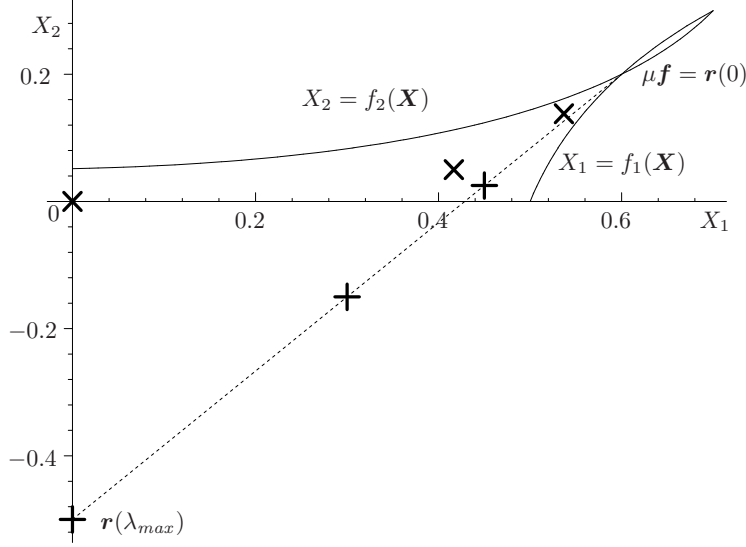


FIG. 5.1. Illustration of Lemma 5.7: The points (shape: +) on the ray  $r$  along a cone vector are lower bounds on the Newton approximants (shape:  $\times$ ).

□

The following proposition guarantees a convergence order of the Newton sequence in terms of a cone vector.

PROPOSITION 5.8. Let  $\mathbf{d}$  be a cone vector of an SPP  $\mathbf{f}$  and let  $\lambda_{max} = \max_j \left\{ \frac{\mu \mathbf{f}_j}{d_j} \right\}$  and  $\lambda_{min} = \min_j \left\{ \frac{\mu \mathbf{f}_j}{d_j} \right\}$ . Let  $k_{\mathbf{f}, \mathbf{d}} = \left\lceil \log \frac{\lambda_{max}}{\lambda_{min}} \right\rceil$ . Then  $\beta(k_{\mathbf{f}, \mathbf{d}} + i) \geq i$  for all  $i \in \mathbb{N}$ .

*Proof.* For all  $1 \leq j \leq n$  the following holds.

$$\begin{aligned}
 (\mu \mathbf{f} - \nu^{(k_{\mathbf{f}, \mathbf{d}} + i)})_j &\leq 2^{-(k_{\mathbf{f}, \mathbf{d}} + i)} \lambda_{max} d_j && \text{(Lemma 5.7)} \\
 &\leq \frac{\lambda_{min}}{\lambda_{max}} 2^{-i} \lambda_{max} d_j && \text{(def. of } k_{\mathbf{f}, \mathbf{d}}) \\
 &= \lambda_{min} d_j \cdot 2^{-i} \\
 &\leq \mu \mathbf{f}_j \cdot 2^{-i} && \text{(def. of } \lambda_{min})
 \end{aligned}$$

Hence,  $\nu^{(k_{\mathbf{f}, \mathbf{d}} + i)}$  has  $i$  valid bits of  $\mu \mathbf{f}$ . □

**5.3. Convergence Speed Independent from Cone Vectors.** The convergence order provided by Proposition 5.8 depends on a cone vector  $\mathbf{d}$ . While Proposition 5.4 guarantees the existence of a cone vector for scSPPs, it does not give any information on the magnitude of its components. So we do not have any bound yet on the “threshold”  $k_{\mathbf{f}, \mathbf{d}}$  from Proposition 5.8. The following theorem solves this problem.

THEOREM 5.9. Let  $\mathbf{f}$  be a quadratic scSPP. Let  $c_{min}$  be the smallest nonzero coefficient of  $\mathbf{f}$  and let  $\mu_{min}$  and  $\mu_{max}$  be the minimal and maximal component of  $\mu \mathbf{f}$ , respectively. Let

$$k_{\mathbf{f}} = \left\lceil \log \frac{\mu_{max}}{\mu_{min} \cdot (c_{min} \cdot \min\{\mu_{min}, 1\})^n} \right\rceil.$$

Then

$$\beta(k_{\mathbf{f}} + i) \geq i \text{ for all } i \in \mathbb{N}.$$

Before we prove Theorem 5.9 we give an example.

EXAMPLE 5.10. *As an example of application of Theorem 5.9 consider the scSPP equation of the back button process of Example 2.9.*

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0.4X_2X_1 + 0.6 \\ 0.3X_1X_2 + 0.4X_3X_2 + 0.3 \\ 0.3X_1X_3 + 0.7 \end{pmatrix}$$

We wish to know if there is a component  $s \in \{1, 2, 3\}$  with  $\mu \mathbf{f}_s = 1$ . Notice that  $\mathbf{f}(\mathbf{1}) = \mathbf{1}$ , so  $\mu \mathbf{f} \leq \mathbf{1}$ . Performing 14 Newton steps (e.g. with Maple) yields an approximation  $\nu^{(14)}$  to  $\mu \mathbf{f}$  with

$$\begin{pmatrix} 0.98 \\ 0.97 \\ 0.992 \end{pmatrix} \leq \nu^{(14)} \leq \begin{pmatrix} 0.99 \\ 0.98 \\ 0.993 \end{pmatrix}.$$

We have  $c_{\min} = 0.3$ . In addition, since Newton's method converges to  $\mu \mathbf{f}$  from below, we know  $\mu_{\min} \geq 0.97$ . Moreover,  $\mu_{\max} \leq 1$ , as  $\mathbf{1} = \mathbf{f}(\mathbf{1})$  and so  $\mu \mathbf{f} \leq \mathbf{1}$ . Hence  $k_{\mathbf{f}} \leq \left\lceil \log \frac{1}{0.97 \cdot (0.3 \cdot 0.97)^3} \right\rceil = 6$ . Theorem 5.9 then implies that  $\nu^{(14)}$  has 8 valid bits of  $\mu \mathbf{f}$ . As  $\mu \mathbf{f} \leq \mathbf{1}$ , the absolute errors are bounded by the relative errors, and since  $2^{-8} \leq 0.004$  we know:

$$\mu \mathbf{f} \leq \nu^{(14)} + \begin{pmatrix} 2^{-8} \\ 2^{-8} \\ 2^{-8} \end{pmatrix} \leq \begin{pmatrix} 0.994 \\ 0.984 \\ 0.997 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So Theorem 5.9 yields a proof that  $\mu \mathbf{f}_s < 1$  for all three components  $s$ .

Notice also that the Newton sequence converges much faster than the Kleene sequence  $(\kappa^{(k)})_{k \in \mathbb{N}}$ . We have  $\kappa^{(14)} \prec (0.89, 0.83, 0.96)^\top$ , so  $\kappa^{(14)}$  has no more than 4 valid bits in any component, whereas  $\nu^{(14)}$  has, in fact, more than 30 valid bits in each component.

For the proof of Theorem 5.9 we need the following lemma.

LEMMA 5.11. *Let  $\mathbf{d}$  be a cone vector of a quadratic scSPP  $\mathbf{f}$ . Let  $c_{\min}$  be the smallest nonzero coefficient of  $\mathbf{f}$  and  $\mu_{\min}$  the minimal component of  $\mu \mathbf{f}$ . Let  $d_{\min}$  and  $d_{\max}$  be the smallest and the largest component of  $\mathbf{d}$ , respectively. Then*

$$\frac{d_{\min}}{d_{\max}} \geq (c_{\min} \cdot \min\{\mu_{\min}, 1\})^n.$$

*Proof.* In what follows we shorten  $\mu \mathbf{f}$  to  $\boldsymbol{\mu}$ . Let w.l.o.g.  $d_1 = d_{\max}$  and  $d_n = d_{\min}$ . We claim the existence of indices  $s, t$  with  $1 \leq s, t \leq n$  such that  $\mathbf{f}'_{st}(\boldsymbol{\mu}) \neq 0$  and

$$\frac{d_{\min}}{d_{\max}} \geq \left( \frac{d_s}{d_t} \right)^n. \quad (5.1)$$

To prove that such  $s, t$  exist, we use the fact that  $\mathbf{f}$  is strongly connected, i.e., that there is a sequence  $1 = r_1, r_2, \dots, r_q = n$  with  $q \leq n$  such that  $\mathbf{f}'_{r_{j+1}r_j}(\mathbf{X})$  is not

constant zero. As  $\boldsymbol{\mu} \succ \mathbf{0}$ , we have  $\mathbf{f}'_{r_{j+1}r_j}(\boldsymbol{\mu}) \neq 0$ . Furthermore

$$\begin{aligned} \frac{d_1}{d_n} &= \frac{d_{r_1}}{d_{r_2}} \cdots \frac{d_{r_{q-1}}}{d_{r_q}}, \text{ and so} \\ \log \frac{d_1}{d_n} &= \log \frac{d_{r_1}}{d_{r_2}} + \cdots + \log \frac{d_{r_{q-1}}}{d_{r_q}}. \end{aligned}$$

So there must exist a  $j$  such that

$$\begin{aligned} \log \frac{d_1}{d_n} &\leq (q-1) \log \frac{d_{r_j}}{d_{r_{j+1}}} \leq n \log \frac{d_{r_j}}{d_{r_{j+1}}}, \text{ and so} \\ \frac{d_n}{d_1} &\geq \left( \frac{d_{r_{j+1}}}{d_{r_j}} \right)^n. \end{aligned}$$

Hence one can choose  $s = r_{j+1}$  and  $t = r_j$ .

As  $\mathbf{d}$  is a cone vector we have  $\mathbf{f}'(\boldsymbol{\mu})\mathbf{d} \leq \mathbf{d}$  and thus  $\mathbf{f}'_{st}(\boldsymbol{\mu})d_t \leq d_s$ . Hence

$$\mathbf{f}'_{st}(\boldsymbol{\mu}) \leq \frac{d_s}{d_t}. \quad (5.2)$$

On the other hand, since  $\mathbf{f}$  is quadratic,  $\mathbf{f}'$  is a linear mapping such that

$$\mathbf{f}'_{st}(\boldsymbol{\mu}) = 2(b_1 \cdot \mu_1 + \cdots + b_n \cdot \mu_n) + \ell$$

where  $b_1, \dots, b_n$  and  $\ell$  are coefficients of quadratic, respectively linear, monomials of  $\mathbf{f}$ . As  $\mathbf{f}'_{st}(\boldsymbol{\mu}) \neq 0$ , at least one of these coefficients must be nonzero and so greater than or equal to  $c_{min}$ . It follows  $\mathbf{f}'_{st}(\boldsymbol{\mu}) \geq c_{min} \cdot \min\{\mu_{min}, 1\}$ . So we have

$$\begin{aligned} (c_{min} \cdot \min\{\mu_{min}, 1\})^n &\leq (\mathbf{f}'_{st}(\boldsymbol{\mu}))^n \\ &\leq \left( \frac{d_s}{d_t} \right)^n && \text{(by (5.2))} \\ &\leq \frac{d_{min}}{d_{max}} && \text{(by (5.1)).} \end{aligned}$$

□

Now we can prove Theorem 5.9.

*Proof.* [of Theorem 5.9] By Proposition 5.4,  $\mathbf{f}$  has a cone vector  $\mathbf{d}$ . Let  $d_{max} = \max_j \{d_j\}$  and  $d_{min} = \min_j \{d_j\}$  and  $\lambda_{max} = \max_j \left\{ \frac{\mu \mathbf{f}_j}{d_j} \right\}$  and  $\lambda_{min} = \min_j \left\{ \frac{\mu \mathbf{f}_j}{d_j} \right\}$ . We have:

$$\begin{aligned} \frac{\lambda_{max}}{\lambda_{min}} &\leq \frac{\mu_{max} \cdot d_{max}}{\mu_{min} \cdot d_{min}} && \text{(as } \lambda_{max} \leq \frac{d_{max}}{\mu_{min}} \text{ and } \lambda_{min} \geq \frac{d_{min}}{\mu_{max}}) \\ &\leq \frac{\mu_{max}}{\mu_{min} \cdot (c_{min} \cdot \min\{\mu_{min}, 1\})^n} && \text{(Lemma 5.11).} \end{aligned}$$

So the statement follows with Proposition 5.8. □

The following consequence of Theorem 5.9 removes some of the parameters on which the  $k_{\mathbf{f}}$  from Theorem 5.9 depends.

**THEOREM 5.12.** *Let  $\mathbf{f}$  be a quadratic scSPP, let  $\mu_{min}$  and  $\mu_{max}$  be the minimal and maximal component of  $\mu \mathbf{f}$ , respectively, and let the coefficients of  $\mathbf{f}$  be given as ratios of  $m$ -bit integers. Then*

$$\beta(k_{\mathbf{f}} + i) \geq i \text{ for all } i \in \mathbb{N}$$

holds for any of the following choices of  $k_{\mathbf{f}}$ .

1.  $\lceil 4mn + 3n \max\{0, -\log \mu_{\min}\} \rceil$ ;
2.  $4mn2^n$ ;
3.  $7mn$  whenever  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$ ;
4.  $2mn + m$  whenever both  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$  and  $\mu_{\max} \leq 1$ .

Items 3. and 4. of Theorem 5.12 apply in particular to termination SPPs of strict pPDAs (§ 2.4), i.e., they satisfy  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$  and  $\mu_{\max} \leq 1$ .

To prove Theorem 5.12 we need some relations between the parameters of  $\mathbf{f}$ . We collect them in the following lemma.

LEMMA 5.13. *Let  $\mathbf{f}$  be a quadratic scSPP. With the terminology of Theorem 5.9 and Theorem 5.12 the following relations hold.*

1.  $c_{\min} \geq 2^{-m}$ .
2. If  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$  then  $\mu_{\min} \geq c_{\min}$ .
3. If  $c_{\min} > 1$  then  $\mu_{\min} > 1$ .
4. If  $c_{\min} \leq 1$  then  $\mu_{\min} \geq c_{\min}^{2^n - 1}$ .
5. If  $\mathbf{f}$  is strictly quadratic, i.e. nonlinear, then the following inequalities hold:  
 $c_{\min} \leq 1$  and  $\mu_{\max} \cdot c_{\min}^{3n-2} \cdot \min\{\mu_{\min}^{2n-2}, 1\} \leq 1$ .

*Proof.* We show the relations in turn.

1. The smallest nonzero coefficient representable as a ratio of  $m$ -bit numbers is  $\frac{1}{2^m}$ .
2. As  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$ , in all components  $i$  there is a nonzero coefficient  $c_i$  such that  $f_i(\mathbf{0}) = c_i$ . We have  $\mu\mathbf{f} \geq \mathbf{f}(\mathbf{0})$ , so  $\mu f_i \geq f_i(\mathbf{0}) = c_i \geq c_{\min} > 0$  holds for all  $i$ . Hence  $\mu_{\min} > 0$ .
3. Let  $c_{\min} > 1$ . Recall the Kleene sequence  $(\kappa^{(k)})_{k \in \mathbb{N}}$  with  $\kappa^{(k)} = \mathbf{f}^k(\mathbf{0})$ . We first show by induction on  $k$  that for all  $k \in \mathbb{N}$  and all components  $i$  either  $\kappa_i^{(k)} = 0$  holds or  $\kappa_i^{(k)} > 1$ . For the induction base we have  $\kappa^{(0)} = \mathbf{0}$ . Let  $k \geq 0$ . Then  $\kappa_i^{(k+1)} = f_i(\kappa^{(k)})$  is a sum of products of numbers which are either coefficients of  $\mathbf{f}$  (and hence by assumption greater than 1) or  $\kappa_j^{(k)}$  for some  $j$ . By induction,  $\kappa_j^{(k)}$  is either 0 or greater than 1. So,  $\kappa_i^{(k+1)}$  must be 0 or greater than 1.  
 By Theorem 2.4, the Kleene sequence converges to  $\mu\mathbf{f}$ . As  $\mathbf{f}$  is clean, we have  $\mu\mathbf{f} \succ \mathbf{0}$ , and so there is a  $k \in \mathbb{N}$  such that  $\kappa^{(k)} \succ \mathbf{1}$ . The statement follows with  $\mu\mathbf{f} \geq \kappa^{(k)}$ .
4. Let  $c_{\min} \leq 1$ . We prove the following stronger statement by induction on  $k$ : For every  $k$  with  $0 \leq k \leq n$  there is a set  $S_k \subseteq \{1, \dots, n\}$ ,  $|S_k| = k$ , such that  $\mu\mathbf{f}_s \geq c_{\min}^{2^k - 1}$  holds for all  $s \in S_k$ . The induction base ( $k = 0$ ) is trivial. Let  $k \geq 0$ . Consider the SPP  $\widehat{\mathbf{f}}(\mathbf{X}_{\{1, \dots, n\} \setminus S_k})$  that is obtained from  $\mathbf{f}(\mathbf{X})$  by removing the  $S_k$ -components from  $\mathbf{f}$  and replacing every  $S_k$ -variable in the polynomials by the corresponding component of  $\mu\mathbf{f}$ . Clearly,  $\mu\widehat{\mathbf{f}} = (\mu\mathbf{f})_{\{1, \dots, n\} \setminus S_k}$ . By induction, the smallest nonzero coefficient  $\widehat{c}_{\min}$  of  $\widehat{\mathbf{f}}$  satisfies  $\widehat{c}_{\min} \geq c_{\min}(c_{\min}^{2^k - 1})^2 = c_{\min}^{2^{k+1} - 1}$ . Pick a component  $i$  with  $\widehat{f}_i(\mathbf{0}) > 0$ . Then  $\mu\widehat{\mathbf{f}}_i \geq \widehat{f}_i(\mathbf{0}) \geq \widehat{c}_{\min} \geq c_{\min}^{2^{k+1} - 1}$ . So set  $S_{k+1} := S_k \cup \{i\}$ .
5. Let w.l.o.g.  $\mu_{\max} = \mu\mathbf{f}_1$ . The proof is based on the idea that  $X_1$  indirectly depends quadratically on itself. More precisely, as  $\mathbf{f}$  is strongly connected and strictly quadratic, component 1 depends (indirectly) on some component, say  $i_r$ , such that  $f_{i_r}$  contains a degree-2-monomial. The variables in that monomial, in turn, depend on  $X_1$ . This gives an inequality of the form  $\mu\mathbf{f}_1 \geq C \cdot \mu\mathbf{f}_1^2$ , implying  $\mu\mathbf{f}_1 \cdot C \leq 1$ .  
 We give the details in the following. As  $\mathbf{f}$  is strongly connected and strictly



quadratic there exists a sequence of variables  $X_{i_1}, \dots, X_{i_r}$  and a sequence of monomials  $m_{i_1}, \dots, m_{i_r}$  ( $1 \leq r \leq n$ ) with the following properties:

- $X_{i_1} = X_1$ ,
- $m_{i_u}$  is a monomial appearing in  $f_{i_u}$  ( $1 \leq u \leq r$ ),
- $m_{i_u} = c_{i_u} \cdot X_{i_{u+1}}$  ( $1 \leq u \leq r$ ),
- $m_{i_r} = c_{i_r} \cdot X_{j_1} \cdot X_{k_1}$  for some variables  $X_{j_1}, X_{k_1}$ .

Notice that

$$\begin{aligned} \mu_{max} = \mu \mathbf{f}_1 &\geq c_{i_1} \cdot \dots \cdot c_{i_r} \cdot \mu \mathbf{f}_{j_1} \cdot \mu \mathbf{f}_{k_1} \\ &\geq \min(c_{min}^n, 1) \cdot \mu \mathbf{f}_{j_1} \cdot \mu \mathbf{f}_{k_1}. \end{aligned} \quad (5.3)$$

Again using that  $\mathbf{f}$  is strongly connected, there exists a sequence of variables  $X_{j_1}, \dots, X_{j_s}$  and a sequence of monomials  $m_{j_1}, \dots, m_{j_{s-1}}$  ( $1 \leq s \leq n$ ) with the following properties:

- $X_{j_s} = X_1$ ,
- $m_{j_u}$  is a monomial appearing in  $f_{j_u}$  ( $1 \leq u \leq s-1$ ),
- $m_{j_u} = c_{j_u} \cdot X_{j_{u+1}}$  or  $m_{j_u} = c_{j_u} \cdot X_{j_{u+1}} \cdot X_{j'_{u+1}}$  for some variable  $X_{j'_{u+1}}$  ( $1 \leq u \leq s-1$ ).

Notice that

$$\begin{aligned} \mu \mathbf{f}_{j_1} &\geq c_{j_1} \cdot \dots \cdot c_{j_{s-1}} \cdot \min(\mu_{min}^{s-1}, 1) \cdot \mu \mathbf{f}_1 \\ &\geq \min(c_{min}^{n-1}, 1) \cdot \min(\mu_{min}^{n-1}, 1) \cdot \mu \mathbf{f}_1. \end{aligned} \quad (5.4)$$

Similarly, there exists a sequence of variables  $X_{k_1}, \dots, X_{k_t}$  ( $1 \leq t \leq n$ ) with  $X_{k_t} = X_1$  showing

$$\mu \mathbf{f}_{k_1} \geq \min(c_{min}^{n-1}, 1) \cdot \min(\mu_{min}^{n-1}, 1) \cdot \mu \mathbf{f}_1. \quad (5.5)$$

Combining (5.3) with (5.4) and (5.5) yields

$$\mu_{max} \geq \min(c_{min}^{3n-2}, 1) \cdot \min(\mu_{min}^{2n-2}, 1) \cdot \mu_{max}^2,$$

or

$$\mu_{max} \cdot \min(c_{min}^{3n-2}, 1) \cdot \min(\mu_{min}^{2n-2}, 1) \leq 1. \quad (5.6)$$

Now it suffices to show  $c_{min} \leq 1$ . Assume for a contradiction  $c_{min} > 1$ . Then, by statement 3.,  $\mu_{min} > 1$ . Plugging this into (5.6) yields  $\mu_{max} \leq 1$ . This implies  $\mu_{max} < \mu_{min}$ , contradicting the definition of  $\mu_{max}$  and  $\mu_{min}$ .

□

Now we are ready to prove Theorem 5.12.

*Proof.* [of Theorem 5.12]

1. First we check the case where  $\mathbf{f}$  is linear, i.e., all polynomials  $f_i$  have degree at most 1. In this case, Newton's method reaches  $\mu \mathbf{f}$  after one iteration, so the statement holds. Consequently, we can assume in the following that  $\mathbf{f}$  is strictly quadratic, meaning that  $\mathbf{f}$  is quadratic and there is a polynomial in  $\mathbf{f}$  of degree 2.

By Theorem 5.9 it suffices to show

$$\log \frac{\mu_{max}}{\mu_{min} \cdot c_{min}^n \cdot \min\{\mu_{min}^n, 1\}} \leq 4mn + 3n \max\{0, -\log \mu_{min}\}.$$

We have

$$\begin{aligned} & \log \frac{\mu_{max}}{\mu_{min} \cdot c_{min}^n \cdot \min\{\mu_{min}^n, 1\}} \\ & \leq \log \frac{1}{c_{min}^{4n-2} \cdot \min\{\mu_{min}^{3n-1}, 1\}} \quad (\text{Lemma 5.13.5.}) \\ & \leq 4n \cdot \log \frac{1}{c_{min}} - \log(\min\{\mu_{min}^{3n-1}, 1\}) \quad (\text{Lemma 5.13.5.: } c_{min} \leq 1) \\ & \leq 4mn - \log(\min\{\mu_{min}^{3n-1}, 1\}) \quad (\text{Lemma 5.13.1.}) \end{aligned}$$

If  $\mu_{min} \geq 1$  we have  $-\log(\min\{\mu_{min}^{3n-1}, 1\}) \leq 0$ , so we are done in this case. If  $\mu_{min} \leq 1$  we have  $-\log(\min\{\mu_{min}^{3n-1}, 1\}) = -(3n-1)\log \mu_{min} \leq 3n \cdot (-\log \mu_{min})$ .

2. By statement 1. of this corollary, it suffices to show that  $4mn + 3n \max\{0, -\log \mu_{min}\} \leq 4mn2^n$ . This inequality obviously holds if  $\mu_{min} \geq 1$ . So let  $\mu_{min} \leq 1$ . Then, by Lemma 5.13.3.,  $c_{min} \leq 1$ . Hence, by Lemma 5.13 parts 4. and 1.,  $\mu_{min} \geq c_{min}^{2^n-1} \geq 2^{-m(2^n-1)}$ . So we have an upper bound on  $-\log \mu_{min}$  with  $-\log \mu_{min} \leq m(2^n-1)$  and get:

$$\begin{aligned} 4mn + 3n \max\{0, -\log \mu_{min}\} & \leq 4mn + 3nm(2^n-1) \\ & \leq 4mn + 4nm(2^n-1) = 4mn2^n \end{aligned}$$

3. Let  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$ . By statement 1. of this corollary it suffices to show that  $4mn + 3n \max\{0, -\log \mu_{min}\} \leq 7mn$  holds. By Lemma 5.13 parts 2. and 1., we have  $\mu_{min} \geq c_{min} \geq 2^{-m}$ , so  $-\log \mu_{min} \leq m$ . Hence,  $4mn + 3n \max\{0, -\log \mu_{min}\} \leq 4mn + 3nm = 7mn$ .
4. Let  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$  and  $\mu_{max} \leq 1$ . By Theorem 5.9 it suffices to show that  $\log \frac{\mu_{max}}{\mu_{min} \cdot c_{min}^n \cdot \min\{\mu_{min}^n, 1\}} \leq 2mn + m$ . We have:

$$\begin{aligned} & \log \frac{\mu_{max}}{\mu_{min} \cdot c_{min}^n \cdot \min\{\mu_{min}^n, 1\}} \\ & \leq -n \log c_{min} - (n+1) \log \mu_{min} \quad (\text{as } \mu_{min} \leq \mu_{max} \leq 1) \\ & \leq -(2n+1) \log c_{min} \quad (\text{Lemma 5.13.2.}) \\ & \leq 2mn + m \quad (\text{Lemma 5.13.1.}) \end{aligned}$$

□

**5.4. Upper Bounds on the Least Fixed Point Via Newton Approximants.** By Theorem 4.8 each Newton approximant  $\nu^{(k)}$  is a lower bound on  $\mu \mathbf{f}$ . Theorem 5.9 and Theorem 5.12 give us upper bounds on the error  $\Delta^{(k)} := \mu \mathbf{f} - \nu^{(k)}$ . Those bounds can directly transformed into upper bounds on  $\mu \mathbf{f}$ , as  $\mu \mathbf{f} = \nu^{(k)} + \Delta^{(k)}$ , cf. Example 5.10.

Theorem 5.9 and Theorem 5.12 allow to compute bounds on  $\Delta^{(k)}$  even before the Newton iteration has been started. However, this may be more than we actually need. In practice, we may wish to use an iterative method that yields guaranteed lower *and*

upper bounds on  $\mu\mathbf{f}$  that improve during the iteration. The following theorem and its corollary can be used to this end.

**THEOREM 5.14.** *Let  $\mathbf{f}$  be a quadratic scSPP. Let  $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{f}$  and  $\mathbf{x} \leq \mathbf{f}(\mathbf{x})$  such that  $\mathbf{f}'(\mathbf{x})^*$  exists. Let  $c_{\min}$  be the smallest nonzero coefficient of  $\mathbf{f}$  and  $\mu_{\min}$  the minimal component of  $\mu\mathbf{f}$ . Then*

$$\frac{\|\mathcal{N}(\mathbf{x}) - \mathbf{x}\|_{\infty}}{\|\mu\mathbf{f} - \mathcal{N}(\mathbf{x})\|_{\infty}} \geq (c_{\min} \cdot \min\{\mu_{\min}, 1\})^n .$$

We prove Theorem 5.14 at the end of the section. The theorem can be applied to the Newton approximants:

**THEOREM 5.15.** *Let  $\mathbf{f}$  be a quadratic scSPP. Let  $c_{\min}$  be the smallest nonzero coefficient of  $\mathbf{f}$  and  $\mu_{\min}$  the minimal component of  $\mu\mathbf{f}$ . For all Newton approximants  $\nu^{(k)}$  with  $\nu^{(k)} \succ \mathbf{0}$ , let  $\nu_{\min}^{(k)}$  be the smallest coefficient of  $\nu^{(k)}$ . Then*

$$\nu^{(k)} \leq \mu\mathbf{f} \leq \nu^{(k)} + \left[ \frac{\|\nu^{(k)} - \nu^{(k-1)}\|_{\infty}}{(c_{\min} \cdot \min\{\nu_{\min}^{(k)}, 1\})^n} \right]$$

where  $[s]$  denotes the vector  $\mathbf{x}$  with  $x_j = s$  for all  $1 \leq j \leq n$ .

*Proof.* [of Theorem 5.15] Theorem 5.14 applies, due to Theorem 4.8, to the Newton approximants with  $\mathbf{x} = \nu^{(k-1)}$ . So we get

$$\begin{aligned} \|\mu\mathbf{f} - \nu^{(k)}\|_{\infty} &\leq \frac{\|\nu^{(k)} - \nu^{(k-1)}\|_{\infty}}{(c_{\min} \cdot \min\{\mu_{\min}, 1\})^n} \\ &\leq \frac{\|\nu^{(k)} - \nu^{(k-1)}\|_{\infty}}{(c_{\min} \cdot \min\{\nu_{\min}^{(k)}, 1\})^n} \quad (\text{as } \nu^{(k)} \leq \mu\mathbf{f}) . \end{aligned}$$

Hence the statement follows from  $\nu^{(k)} \leq \mu\mathbf{f}$ .  $\square$

**EXAMPLE 5.16.** *Consider again the equation  $\mathbf{X} = \mathbf{f}(\mathbf{X})$  from Examples 2.9 and 5.10:*

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0.4X_2X_1 + 0.6 \\ 0.3X_1X_2 + 0.4X_3X_2 + 0.3 \\ 0.3X_1X_3 + 0.7 \end{pmatrix}$$

Again we wish to verify that there is no component  $s \in \{1, 2, 3\}$  with  $\mu\mathbf{f}_s = 1$ . Performing 10 Newton steps yields an approximation  $\nu^{(10)}$  to  $\mu\mathbf{f}$  with

$$\begin{pmatrix} 0.9828 \\ 0.9738 \\ 0.9926 \end{pmatrix} \prec \nu^{(10)} \prec \begin{pmatrix} 0.9829 \\ 0.9739 \\ 0.9927 \end{pmatrix} .$$

Further, it holds  $\|\nu^{(10)} - \nu^{(9)}\|_{\infty} \leq 2 \cdot 10^{-6}$ . So we have

$$\frac{\|\nu^{(10)} - \nu^{(9)}\|_{\infty}}{(c_{\min} \cdot \min\{\nu_{\min}^{(10)}, 1\})^3} \leq \frac{2 \cdot 10^{-6}}{(0.3 \cdot 0.97)^3} \leq 0.00009$$

and hence by Theorem 5.15

$$\nu^{(10)} \leq \mu\mathbf{f} \leq \nu^{(10)} + [0.00009] \leq \begin{pmatrix} 0.983 \\ 0.974 \\ 0.993 \end{pmatrix}$$

In particular we know that  $\mu \mathbf{f}_s < 1$  for all three components  $s$ .

EXAMPLE 5.17. Consider again the SPP  $\mathbf{f}$  from Example 5.16. Setting

$$\mathbf{u}^{(k)} := \boldsymbol{\nu}^{(k)} + \left[ \frac{\|\boldsymbol{\nu}^{(k)} - \boldsymbol{\nu}^{(k-1)}\|_\infty}{\left(0.3 \cdot \nu_{\min}^{(k)}\right)^3} \right],$$

Theorem 5.15 guarantees

$$\boldsymbol{\nu}^{(k)} \leq \mu \mathbf{f} \leq \mathbf{u}^{(k)}.$$

Let us measure the tightness of the bounds  $\boldsymbol{\nu}^{(k)}$  and  $\mathbf{u}^{(k)}$  on  $\mu \mathbf{f}$  in the first component. Let

$$\begin{aligned} p_{\text{lower}}(k) &:= -\log_2(\mu \mathbf{f}_1 - \nu_1^{(k)}) && \text{and} \\ p_{\text{upper}}(k) &:= -\log_2(u_1^{(k)} - \mu \mathbf{f}_1). \end{aligned}$$

Roughly speaking,  $\nu_1^{(k)}$  and  $u_1^{(k)}$  have  $p_{\text{lower}}(k)$  and  $p_{\text{upper}}(k)$  valid bits of  $\mu \mathbf{f}_1$ , respectively. Figure 5.2 shows  $p_{\text{lower}}(k)$  and  $p_{\text{upper}}(k)$  for  $k \in \{1, \dots, 11\}$ .

It can be seen that the slope of  $p_{\text{lower}}(k)$  is approximately 1 for  $k = 2, \dots, 6$ . This corresponds to the linear convergence of Newton's method according to Theorem 5.9. Since  $\text{Id} - \mathbf{f}'(\mu \mathbf{f})$  is non-singular<sup>4</sup>, Newton's method actually has, asymptotically, an exponential convergence order, cf. Theorem 4.13. This behavior can be observed in Figure 5.2 for  $k \geq 7$ . For  $p_{\text{upper}}$ , we roughly have (using  $\boldsymbol{\nu}^{(k)} \approx \mu \mathbf{f}$ ):

$$p_{\text{upper}}(k) \approx p_{\text{lower}}(k-1) + \log \left(0.3 \cdot \nu_{\min}^{(k)}\right)^3 \approx p_{\text{lower}}(k-1) - 5.$$

The proof of Theorem 5.14 uses similar techniques as the proof of Theorem 5.9, in particular Lemma 5.11.

*Proof.* [of Theorem 5.14] By Proposition 5.4,  $\mathbf{f}$  has a cone vector  $\mathbf{d}$ . Let  $d_{\min}$  and  $d_{\max}$  be the smallest and the largest component of  $\mathbf{d}$ , respectively. Let  $\lambda_{\max} := \max_j \left\{ \frac{\mu \mathbf{f}_j - x_j}{d_j} \right\}$ , and let w.l.o.g.  $\lambda_{\max} = \frac{\mu \mathbf{f}_1 - x_1}{d_1}$ . We have  $\mathbf{x} \geq \mu \mathbf{f} - \lambda_{\max} \mathbf{d}$ , so we can apply Lemma 5.6 to obtain  $\mathcal{N}(\mathbf{x}) \geq \mu \mathbf{f} - \frac{1}{2} \lambda_{\max} \mathbf{d}$ . Thus

$$\|\mathcal{N}(\mathbf{x}) - \mathbf{x}\|_\infty \geq (\mathcal{N}(\mathbf{x}) - \mathbf{x})_1 \geq \mu \mathbf{f}_1 - \frac{1}{2} \lambda_{\max} d_1 - x_1 = \frac{1}{2} \lambda_{\max} d_1 \geq \frac{1}{2} \lambda_{\max} d_{\min}.$$

On the other hand, with Lemma 4.3 we have  $\mathbf{0} \leq \mu \mathbf{f} - \mathcal{N}(\mathbf{x}) \leq \frac{1}{2} \lambda_{\max} \mathbf{d}$  and so  $\|\mu \mathbf{f} - \mathcal{N}(\mathbf{x})\|_\infty \leq \frac{1}{2} \lambda_{\max} d_{\max}$ . Combining those inequalities we obtain

$$\frac{\|\mathcal{N}(\mathbf{x}) - \mathbf{x}\|_\infty}{\|\mu \mathbf{f} - \mathcal{N}(\mathbf{x})\|_\infty} \geq \frac{d_{\min}}{d_{\max}}.$$

Now the statement follows from Lemma 5.11.  $\square$

<sup>4</sup>In fact, the matrix is “almost” singular, with  $\det(\text{Id} - \mathbf{f}'(\mu \mathbf{f})) \approx 0.006$ .

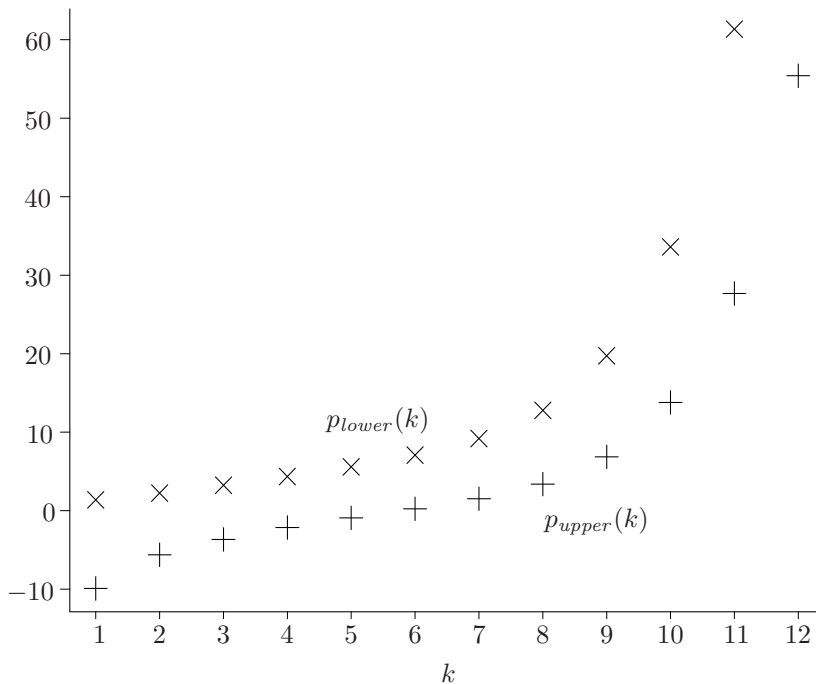


FIG. 5.2. Number of valid bits of the lower (shape:  $\times$ ) and upper (shape:  $+$ ) bounds on  $\mu\mathbf{f}_1$ , see Example 5.17.

**6. General SPPs.** In § 5 we considered *strongly connected* SPPs, see Definition 2.7. However, it is not always guaranteed that the SPP  $\mathbf{f}$  is strongly connected. In this section we analyze the convergence speed of two variants of Newton’s method that both compute approximations of  $\mu\mathbf{f}$ , where  $\mathbf{f}$  is an SPP that is not necessarily strongly connected (“general SPPs”).

The first one was suggested by Etessami and Yannakakis [14] and is called *Decomposed Newton’s Method (DNM)*. It works by running Newton’s method separately on each SCC, see § 6.1. The second one is the regular Newton’s method from § 4. We will analyze its convergence speed in § 6.2.

The reason why we first analyze DNM is that our convergence speed results about Newton’s method for general SPPs (Theorem 6.5) build on our results about DNM (Theorem 6.2). From an efficiency point of view it actually may be advantageous to run Newton’s method separately on each SCC. For those reasons DNM deserves a separate treatment.

**6.1. Convergence Speed of the Decomposed Newton’s Method (DNM).** DNM, originally suggested in [14], works as follows. It starts by using Newton’s method for each bottom SCC, say  $S$ , of the SPP  $\mathbf{f}$ . Then the corresponding variables  $\mathbf{X}_S$  are substituted for the obtained approximation for  $\mu\mathbf{f}_S$ , and the corresponding equations  $\mathbf{X}_S = \mathbf{f}_S(\mathbf{X})$  are removed. The same procedure is then applied to the new bottom SCCs, until all SCCs have been processed.

Etessami and Yannakakis did not provide a particular criterion for the number of Newton iterations to be applied in each SCC. Consequently, they did not analyze the convergence speed of DNM. We will treat those issues in this section, thereby taking advantage of our previous analysis of scSPPs.

We fix a quadratic SPP  $\mathbf{f}$  for this section. We assume that we have already computed the DAG (directed acyclic graph) of SCCs. This can be done in linear time in the size of  $\mathbf{f}$ . To each SCC  $S$  we can associate its *depth*  $t$ : it is the longest path in the DAG of SCCs from  $S$  to a top SCC. Notice that  $0 \leq t \leq n - 1$ . We write  $\text{SCC}(t)$  for the set of SCCs of depth  $t$ . We define the height  $h(\mathbf{f})$  as the largest depth of an SCC and the width  $w(\mathbf{f}) := \max_t |\text{SCC}(t)|$  as the largest number of SCCs of the same depth. Notice that  $\mathbf{f}$  has at most  $(h(\mathbf{f}) + 1) \cdot w(\mathbf{f})$  SCCs. Further we define the component sets  $[t] := \bigcup_{S \in \text{SCC}(t)} S$  and  $[> t] := \bigcup_{t' > t} [t']$  and similarly  $[< t]$ .

```

function DNM ( $\mathbf{f}, i$ )    /* The parameter  $i$  controls the precision. */
  for  $t$  from  $h(\mathbf{f})$  downto 0
    forall  $S \in \text{SCC}(t)$           /* for all SCCs  $S$  of depth  $t$  */
       $\rho_S^{(i)} := \mathcal{N}_{\mathbf{f}_S}^{i \cdot 2^t}(\mathbf{0})$     /* perform  $i \cdot 2^t$  Newton iterations */
       $\mathbf{f}_{[< t]} := \mathbf{f}_{[< t]}[S/\rho_S^{(i)}]$     /* apply  $\rho_S^{(i)}$  in the upper SCCs */
  return  $\rho^{(i)}$ 

```

FIG. 6.1. *Decomposed Newton's Method (DNM) for computing an approximation  $\rho^{(i)}$  of  $\mu\mathbf{f}$*

Figure 6.1 shows our version of DNM. We suggest to run Newton's method in each SCC  $S$  for a number of steps that depends (exponentially) on the depth of  $S$  and (linearly) on a parameter  $i$  that controls the precision.

**PROPOSITION 6.1.** *The function  $\text{DNM}(\mathbf{f}, i)$  of Figure 6.1 runs at most  $i \cdot w(\mathbf{f}) \cdot 2^{h(\mathbf{f})+1} \leq i \cdot n \cdot 2^n$  iterations of Newton's method.*

*Proof.* The number of iterations is  $\sum_{t=0}^{h(\mathbf{f})} |\text{SCC}(t)| \cdot i \cdot 2^t$ . This can be estimated as follows.

$$\begin{aligned}
\sum_{t=0}^{h(\mathbf{f})} |\text{SCC}(t)| \cdot i \cdot 2^t &\leq w(\mathbf{f}) \cdot i \cdot \sum_{t=0}^{h(\mathbf{f})} 2^t \\
&\leq w(\mathbf{f}) \cdot i \cdot 2^{h(\mathbf{f})+1} \\
&\leq i \cdot n \cdot 2^n \qquad \qquad \qquad (\text{as } w(\mathbf{f}) \leq n \text{ and } h(\mathbf{f}) < n)
\end{aligned}$$

□

The following theorem states that DNM has linear convergence order.

**THEOREM 6.2.** *Let  $\mathbf{f}$  be a quadratic SPP. Let  $\rho^{(i)}$  denote the result of calling  $\text{DNM}(\mathbf{f}, i)$  (see Figure 6.1). Let  $\beta_\rho$  denote the convergence order of  $(\rho^{(i)})_{i \in \mathbb{N}}$ . Then there is a  $k_\mathbf{f} \in \mathbb{N}$  such that  $\beta_\rho(k_\mathbf{f} + i) \geq i$  for all  $i \in \mathbb{N}$ .*

Theorem 6.2 can be interpreted as follows: Increasing  $i$  by one yields asymptotically at least one additional bit in each component and, by Proposition 6.1, costs at most  $n \cdot 2^n$  additional Newton iterations. Notice that for simplicity we do not take into account here that the cost of performing a Newton step on a single SCC is not uniform, but rather depends on the size of the SCC (e.g. cubically if Gaussian elimination is used for solving the linear systems).

For the proof of Theorem 6.2, let  $\Delta^{(i)}$  denote the error when running DNM with parameter  $i$ , i.e.,  $\Delta^{(i)} := \mu\mathbf{f} - \rho^{(i)}$ . Observe that the error  $\Delta^{(i)}$  can be understood as the sum of two errors:

$$\Delta^{(i)} := \mu\mathbf{f} - \rho^{(i)} = (\mu - \tilde{\mu}^{(i)}) + (\tilde{\mu}^{(i)} - \rho^{(i)}),$$

where  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)} := \mu(\mathbf{f}_{[t]}[[> t]/\boldsymbol{\rho}_{[> t]}^{(i)}])$ , i.e.,  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$  is the least fixed point of  $\mathbf{f}_{[t]}$  after the approximations from the lower SCCs have been applied. So,  $\boldsymbol{\Delta}_{[t]}^{(i)}$  consists of the *propagation error* ( $\mu\mathbf{f}_{[t]} - \tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$ ) (resulting from the error at lower SCCs) and the *approximation error* ( $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)} - \boldsymbol{\rho}_{[t]}^{(i)}$ ) (resulting from the newly added error of Newton's method on level  $t$ ).

The following lemma gives a bound on the propagation error.

LEMMA 6.3 (Propagation error). *There is a constant  $C_{\mathbf{f}} > 0$  such that*

$$\left\| \mu\mathbf{f}_{[t]} - \tilde{\boldsymbol{\mu}}_{[t]} \right\| \leq C_{\mathbf{f}} \cdot \sqrt{\left\| \mu\mathbf{f}_{[> t]} - \boldsymbol{\rho}_{[> t]} \right\|}$$

holds for all  $\boldsymbol{\rho}_{[> t]}$  with  $\mathbf{0} \leq \boldsymbol{\rho}_{[> t]} \leq \mu\mathbf{f}_{[> t]}$ , where  $\tilde{\boldsymbol{\mu}}_{[t]} = \mu(\mathbf{f}_{[t]}[[> t]/\boldsymbol{\rho}_{[> t]}])$ .

Roughly speaking, Lemma 6.3 states that if  $\boldsymbol{\rho}_{[> t]}^{(i)}$  has  $k$  valid bits of  $\mu\mathbf{f}_{[> t]}$ , then  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$  has at least about  $k/2$  valid bits of  $\mu\mathbf{f}_{[t]}$ . In other words, (at most) one half of the valid bits are lost on each level of the DAG due to the propagation error. The proof of Lemma 6.3 is technically involved and, unfortunately, not constructive in that we know nothing about  $C_{\mathbf{f}}$  except for its existence. The proof can be found in Appendix A.

The following lemma gives a bound on the error  $\left\| \boldsymbol{\Delta}_{[t]}^{(i)} \right\|$  on level  $t$ , taking both the propagation error and the approximation error into account.

LEMMA 6.4. *There is a  $C_{\mathbf{f}} > 0$  such that  $\left\| \boldsymbol{\Delta}_{[t]}^{(i)} \right\| \leq 2^{C_{\mathbf{f}} - i \cdot 2^t}$  for all  $i \in \mathbb{N}$ .*

*Proof.* Let  $\tilde{\mathbf{f}}_{[t]}^{(i)} := \mathbf{f}_{[t]}[[> t]/\boldsymbol{\rho}_{[> t]}^{(i)}]$ . Observe that the coefficients of  $\tilde{\mathbf{f}}_{[t]}^{(i)}$  and thus its least fixed point  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$  are monotonically increasing with  $i$ , because  $\boldsymbol{\rho}_{[> t]}^{(i)}$  is monotonically increasing as well. Consider an arbitrary depth  $t$  and choose real numbers  $c_{\min} > 0$  and  $\mu_{\min} > 0$  and an integer  $i_0$  such that, for all  $i \geq i_0$ ,  $c_{\min}$  and  $\mu_{\min}$  are lower bounds on the smallest nonzero coefficient of  $\tilde{\mathbf{f}}_{[t]}^{(i)}$  and the smallest coefficient of  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$ , respectively. Let  $\mu_{\max}$  be the largest component of  $\mu\mathbf{f}_{[t]}$ . Let  $\tilde{k} := \left\lceil n \cdot \log \frac{\mu_{\max}}{c_{\min} \cdot \mu_{\min} \cdot \min\{\mu_{\min}, 1\}} \right\rceil$ . Then it follows from Theorem 5.9 that performing  $\tilde{k} + j$  Newton iterations ( $j \geq 0$ ) on depth  $t$  yields  $j$  valid bits of  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$  for any  $i \geq i_0$ . In particular,  $\tilde{k} + i \cdot 2^t$  Newton iterations give  $i \cdot 2^t$  valid bits of  $\tilde{\boldsymbol{\mu}}_{[t]}^{(i)}$  for any  $i \geq i_0$ . So there exists a constant  $c_1 > 0$  such that, for all  $i \geq i_0$ ,

$$\left\| \tilde{\boldsymbol{\mu}}_{[t]}^{(i)} - \boldsymbol{\rho}_{[t]}^{(i)} \right\| \leq 2^{c_1 - i \cdot 2^t}, \quad (6.1)$$

because DNM (see Figure 6.1) performs  $i \cdot 2^t$  iterations to compute  $\boldsymbol{\rho}_S^{(i)}$  where  $S$  is an SCC of depth  $t$ . Choose  $c_1$  large enough such that Equation (6.1) holds for all  $i \geq 0$  and all depths  $t$ .

Now we can prove the theorem by induction on  $t$ . In the base case ( $t = h(\mathbf{f})$ ) there is no propagation error, so the claim of the lemma follows from (6.1). Let  $t < h(\mathbf{f})$ . Then

$$\begin{aligned} \left\| \boldsymbol{\Delta}_{[t]}^{(i)} \right\| &= \left\| \mu\mathbf{f}_{[t]} - \tilde{\boldsymbol{\mu}}_{[t]}^{(i)} + \tilde{\boldsymbol{\mu}}_{[t]}^{(i)} - \boldsymbol{\rho}_{[t]}^{(i)} \right\| \\ &\leq \left\| \mu\mathbf{f}_{[t]} - \tilde{\boldsymbol{\mu}}_{[t]}^{(i)} \right\| + \left\| \tilde{\boldsymbol{\mu}}_{[t]}^{(i)} - \boldsymbol{\rho}_{[t]}^{(i)} \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \mu \mathbf{f}_{[t]} - \tilde{\boldsymbol{\mu}}_{[t]}^{(i)} \right\| + 2^{c_1 - i \cdot 2^t} && \text{(by (6.1))} \\
&\leq c_2 \cdot \sqrt{\left\| \Delta_{[>t]}^{(i)} \right\|} + 2^{c_1 - i \cdot 2^t} && \text{(Lemma 6.3)} \\
&\leq c_2 \cdot \sqrt{2^{c_3 - i \cdot 2^{t+1}}} + 2^{c_1 - i \cdot 2^t} && \text{(induction hypothesis)} \\
&\leq 2^{c_4 - i \cdot 2^t}
\end{aligned}$$

for some constants  $c_2, c_3, c_4 > 0$ .  $\square$

Now Theorem 6.2 follows easily.

*Proof.* [of Theorem 6.2] From Lemma 6.4 we deduce that for each component  $j \in [t]$  there is a  $c_j$  such that

$$(\mu \mathbf{f}_j - \rho_j^{(i)}) / \mu \mathbf{f}_j \leq 2^{c_j - i \cdot 2^t} \leq 2^{c_j - i}.$$

Let  $k_{\mathbf{f}} \geq c_j$  for all  $1 \leq j \leq n$ . Then

$$(\mu \mathbf{f}_j - \rho_j^{(i+k_{\mathbf{f}})}) / \mu \mathbf{f}_j \leq 2^{c_j - (i+k_{\mathbf{f}})} \leq 2^{-i}.$$

$\square$

Notice that, unfortunately, we cannot give a bound on  $k_{\mathbf{f}}$ , mainly because Lemma 6.3 does not provide a bound on  $C_{\mathbf{f}}$ .

**6.2. Convergence Speed of Newton's Method.** We use the Theorem 6.2 to prove the following theorem for the regular (i.e. not decomposed) Newton sequence  $(\boldsymbol{\nu}^{(i)})_{i \in \mathbb{N}}$ .

**THEOREM 6.5.** *Let  $\mathbf{f}$  be a quadratic SPP. There is a threshold  $k_{\mathbf{f}} \in \mathbb{N}$  such that  $\beta(k_{\mathbf{f}} + i \cdot n \cdot 2^n) \geq \beta(k_{\mathbf{f}} + i \cdot (h(\mathbf{f}) + 1) \cdot 2^{h(\mathbf{f})}) \geq i$  for all  $i \in \mathbb{N}$ .*

In the rest of the section we prove this theorem by a sequence of lemmata. The following lemma states that a Newton step is not faster on an SCC, if the values of the lower SCCs are fixed.

**LEMMA 6.6.** *Let  $\mathbf{f}$  be an SPP. Let  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{f}(\mathbf{x}) \leq \mu \mathbf{f}$  such that  $\mathbf{f}'(\mathbf{x})^*$  exists. Let  $S$  be an SCC of  $\mathbf{f}$  and let  $L$  denote the set of components that are not in  $S$ , but on which a variable in  $S$  depends. Then  $(\mathcal{N}_{\mathbf{f}}(\mathbf{x}))_S \geq \mathcal{N}_{\mathbf{f}_S[L/\mathbf{x}_L]}(\mathbf{x}_S)$ .*

*Proof.*

$$\begin{aligned}
(\mathcal{N}_{\mathbf{f}}(\mathbf{x}))_S &= (\mathbf{f}'(\mathbf{x})^*(\mathbf{f}(\mathbf{x}) - \mathbf{x}))_S \\
&= \mathbf{f}'(\mathbf{x})_{SS}^*(\mathbf{f}(\mathbf{x}) - \mathbf{x})_S + \mathbf{f}'(\mathbf{x})_{SL}^*(\mathbf{f}(\mathbf{x}) - \mathbf{x})_L \\
&\geq \mathbf{f}'(\mathbf{x})_{SS}^*(\mathbf{f}(\mathbf{x}) - \mathbf{x})_S \\
&= ((\mathbf{f}_S[L/\mathbf{x}_L])'(\mathbf{x}_S))^*(\mathbf{f}_S[L/\mathbf{x}_L](\mathbf{x}_S) - \mathbf{x}_S) \\
&= \mathcal{N}_{\mathbf{f}_S[L/\mathbf{x}_L]}(\mathbf{x}_S)
\end{aligned}$$

$\square$

Recall Lemma 4.9 which states that the Newton operator  $\mathcal{N}$  is monotone. This fact and Lemma 6.6 can be combined to the following lemma stating that  $i \cdot (h(\mathbf{f}) + 1)$  iterations of the regular Newton's method “dominate” a decomposed Newton's method that performs  $i$  Newton steps in each SCC.

**LEMMA 6.7.** *Let  $\tilde{\boldsymbol{\nu}}^{(i)}$  denote the result of a decomposed Newton's method which performs  $i$  iterations of Newton's method in each SCC. Let  $\boldsymbol{\nu}^{(i)}$  denote the result of  $i$  iterations of the regular Newton's method. Then  $\boldsymbol{\nu}^{(i \cdot (h(\mathbf{f}) + 1))} \geq \tilde{\boldsymbol{\nu}}^{(i)}$ .*



*Proof.* Let  $h = h(\mathbf{f})$ . Let  $[t]$  and  $[> t]$  again denote the set of components of depth  $t$  and  $> t$ , respectively. We show by induction on the depth  $t$ :

$$\boldsymbol{\nu}_{[t]}^{(i \cdot (h+1-t))} \geq \tilde{\boldsymbol{\nu}}_{[t]}^{(i)}$$

The induction base ( $t = h$ ) is clear, because for bottom SCCs the two methods are identical. Let now  $t < h$ . Then

$$\begin{aligned} \boldsymbol{\nu}_{[t]}^{(i \cdot (h+1-t))} &= \mathcal{N}_{\mathbf{f}}^i(\boldsymbol{\nu}^{(i \cdot (h-t))})_{[t]} \\ &\geq \mathcal{N}_{\mathbf{f}_{[t][>t]/\boldsymbol{\nu}_{[>t]}^{(i \cdot (h-t))}}}^i(\boldsymbol{\nu}_{[t]}^{(i \cdot (h-t))}) && \text{(Lemma 6.6)} \\ &\geq \mathcal{N}_{\mathbf{f}_{[t][>t]/\tilde{\boldsymbol{\nu}}_{[>t]}^{(i)}}}^i(\boldsymbol{\nu}_{[t]}^{(i \cdot (h-t))}) && \text{(induction hypothesis)} \\ &\geq \mathcal{N}_{\mathbf{f}_{[t][>t]/\tilde{\boldsymbol{\nu}}_{[>t]}^{(i)}}}^i(\mathbf{0}_{[t]}) && \text{(Lemma 4.9)} \\ &= \tilde{\boldsymbol{\nu}}_{[t]}^{(i)} && \text{(definition of } \tilde{\boldsymbol{\nu}}^{(i)}) \end{aligned}$$

Now, the lemma itself follows by using Lemma 4.9 once more.  $\square$

As a side note, observe that above proof of Lemma 6.7 implicitly benefits from the fact that SCCs of the same depth are independent. So, SCCs with the same depth are handled in parallel by the regular Newton's method. Therefore,  $w(\mathbf{f})$ , the width of  $\mathbf{f}$ , is irrelevant here (cf. Proposition 6.1).

Now we can prove Theorem 6.5.

*Proof.* [of Theorem 6.5] Let  $k_2$  be the  $k_{\mathbf{f}}$  of Theorem 6.2, and let  $k_1 = k_2 \cdot (h(\mathbf{f}) + 1) \cdot 2^{h(\mathbf{f})}$ . Then we have

$$\begin{aligned} \boldsymbol{\nu}^{(k_1+i \cdot (h(\mathbf{f})+1) \cdot 2^{h(\mathbf{f})})} &= \boldsymbol{\nu}^{((k_2+i) \cdot (h(\mathbf{f})+1) \cdot 2^{h(\mathbf{f})})} \\ &\geq \tilde{\boldsymbol{\nu}}^{((k_2+i) \cdot 2^{h(\mathbf{f})})} && \text{(Lemma 6.7)} \\ &\geq \boldsymbol{\rho}^{(k_2+i)}, \end{aligned}$$

where the last step follows from the fact that  $\text{DNM}(\mathbf{f}, k_2+i)$  runs at most  $(k_2+i) \cdot 2^{h(\mathbf{f})}$  iterations in every SCC. By Theorem 6.2,  $\boldsymbol{\rho}^{(k_2+i)}$  and hence  $\boldsymbol{\nu}^{(k_1+i \cdot (h(\mathbf{f})+1) \cdot 2^{h(\mathbf{f})})}$  have  $i$  valid bits of  $\mu\mathbf{f}$ . Therefore, Theorem 6.5 holds with  $k_{\mathbf{f}} = k_1$ .  $\square$

**7. Upper Bounds on the Convergence.** In this section we show that the lower bounds on the convergence order of Newton's method that we obtained in the previous section are essentially tight, meaning that an exponential (in  $n$ ) number of iterations may be needed per bit.

More precisely, we expose a family  $(\mathbf{f}^{(n)})_{n \geq 1}$  of SPPs with  $n$  variables, such that more than  $k \cdot 2^{n-1}$  iterations are needed for  $k$  valid bits. Consider the following system.

$$\mathbf{X} = \mathbf{f}^{(n)}(\mathbf{X}) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}X_1^2 \\ \frac{1}{4}X_1^2 + \frac{1}{2}X_1X_2 + \frac{1}{4}X_2^2 \\ \vdots \\ \frac{1}{4}X_{n-1}^2 + \frac{1}{2}X_{n-1}X_n + \frac{1}{4}X_n^2 \end{pmatrix} \quad (7.1)$$

The only solution of (7.1) is  $\mu\mathbf{f}^{(n)} = (1, \dots, 1)^\top$ . Notice that each component of  $\mathbf{f}^{(n)}$  is an SCC. We prove the following theorem.

**THEOREM 7.1.** *The convergence order of Newton's method applied to the SPP  $\mathbf{f}^{(n)}$  from (7.1) (with  $n \geq 2$ ) satisfies*

$$\beta(k \cdot 2^{n-1}) < k \text{ for all } k \in \{1, 2, \dots\}.$$

In particular,  $\beta(2^{n-1}) = 0$ .

*Proof.* We write  $\mathbf{f} := \mathbf{f}^{(n)}$  for simplicity. Let

$$\Delta^{(i)} := \mu\mathbf{f} - \boldsymbol{\nu}^{(i)} = (1, \dots, 1)^\top - \boldsymbol{\nu}^{(i)}.$$

Notice that  $(\nu_1^{(i)})_{i \in \mathbb{N}} = (0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots)$  which is the same sequence as obtained by applying Newton's method to the 1-dimensional system  $X_1 = \frac{1}{2} + \frac{1}{2}X_1^2$ . So we have  $\Delta_1^{(i)} = 2^{-i}$ , i.e., after  $i$  iterations we have exactly  $i$  valid bits in the first component.

We know from Theorem 4.8 that for all  $j$  with  $1 \leq j \leq n-1$  we have  $\nu_{j+1}^{(i)} \leq f_{j+1}(\boldsymbol{\nu}^{(i)}) = \frac{1}{4}(\nu_j^{(i)})^2 + \frac{1}{2}\nu_j^{(i)}\nu_{j+1}^{(i)} + \frac{1}{4}(\nu_{j+1}^{(i)})^2$  and  $\nu_{j+1}^{(i)} \leq 1$ . It follows that  $\nu_{j+1}^{(i)}$  is at most the least solution of  $X_{j+1} = \frac{1}{4}(\nu_j^{(i)})^2 + \frac{1}{2}\nu_j^{(i)}X_{j+1} + \frac{1}{4}(X_{j+1})^2$ , and so  $\Delta_{j+1}^{(i)} \geq 2\sqrt{\Delta_j^{(i)}} - \Delta_j^{(i)} > \sqrt{\Delta_j^{(i)}}$ .

By induction it follows that  $\Delta_{j+1}^{(i)} > (\Delta_1^{(i)})^{2^{-j}}$ . In particular,

$$\Delta_n^{(k \cdot 2^{n-1})} > \left(\Delta_1^{(k \cdot 2^{n-1})}\right)^{2^{-(n-1)}} = 2^{-k \cdot 2^{n-1} \cdot 2^{-(n-1)}} = 2^{-k}.$$

Hence, after  $k \cdot 2^{n-1}$  iterations we have less than  $k$  valid bits.  $\square$

Notice that the proof exploits that an error in the first component gets ‘‘amplified’’ along the DAG of SCCs. One can also show along those lines that computing  $\mu\mathbf{f}$  is an *ill-conditioned* problem: Consider the SPP  $\mathbf{g}^{(n, \varepsilon)}$  obtained from  $\mathbf{f}^{(n)}$  by replacing the first component by  $1 - \varepsilon$  where  $0 \leq \varepsilon < 1$ . If  $\varepsilon = 0$  then  $(\mu\mathbf{g}^{(n, \varepsilon)})_n = 1$ , whereas if  $\varepsilon = \frac{1}{2^{2^n - 1}}$  then  $(\mu\mathbf{g}^{(n, \varepsilon)})_n < \frac{1}{2}$ . In other words, to get 1 bit of precision of  $\mu\mathbf{g}$  one needs exponentially in  $n$  many bits in  $\mathbf{g}$ . Note that this observation is independent from any particular method to compute or approximate the least fixed point.

**8. Geometrical Aspects of SPPs.** As shown in § 4.4 we can assume that  $\mathbf{f}$  consists of quadratic polynomials. For quadratic polynomials the locus of zeros is also called a *quadric surface*, or more commonly *quadric*. Quadrics are one of the most fundamental class of hypersurfaces. It is therefore natural to study the quadrics induced by a quadratic SPP  $\mathbf{f}$ , and how the Newton sequence is connected to these surfaces.

Let us write  $\mathbf{q}$  for  $\mathbf{f} - \mathbf{X}$ . Every component  $q_i$  of  $\mathbf{q}$  is also a quadratic polynomial each defining a quadric denoted by

$$Q_i := \{\mathbf{x} \in \mathbb{R}^n \mid q_i(\mathbf{x}) = f_i(\mathbf{x}) - x_i = 0\}.$$

Finding  $\mu\mathbf{f}$  thus corresponds to finding the least non-negative point of intersection of these  $n$  quadrics  $Q_i$ .

**EXAMPLE 8.1.** *Consider the SPP  $\mathbf{f}$  given by*

$$\mathbf{f}(X, Y) = \left( \begin{array}{c} \frac{1}{2}X^2 + \frac{1}{4}Y^2 + \frac{1}{4} \\ \frac{1}{4}X + \frac{1}{4}XY + \frac{1}{4}Y^2 + \frac{1}{4} \end{array} \right)$$

leading to

$$q_1(X, Y) = \frac{1}{2}X^2 + \frac{1}{4}Y^2 + \frac{1}{4} - X \text{ and } q_2(X, Y) = \frac{1}{4}X + \frac{1}{4}XY + \frac{1}{4}Y^2 + \frac{1}{4} - Y.$$

Using standard techniques from linear algebra one can show that  $q_1$  defines an ellipse while  $q_2$  describes a parabola (see Figure 8.1).

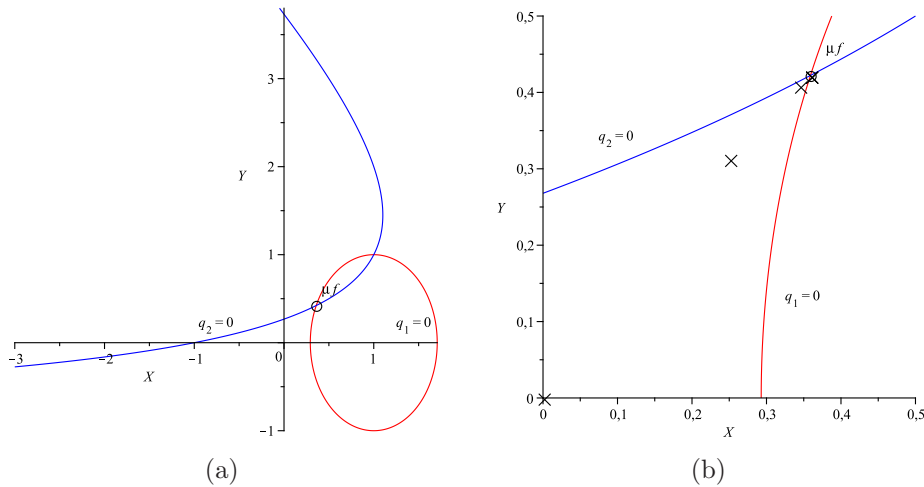


FIG. 8.1. (a) The quadrics induced by the SPP from Example 8.1 with “ $q_1 = 0$ ” an ellipse, and “ $q_2 = 0$ ” a parabola. (b) Close-up view of the region important for determining  $\mu\mathbf{f}$ . The crosses show the Newton approximants of  $\mu\mathbf{f}$ .

Figure 8.1 shows the two quadrics induced by the SPP  $\mathbf{f}$  discussed in the example above. In Figure 8.1 (a) one can recognize one of the two quadrics as an ellipse while the other one is a parabola. In this example the Newton approximants (depicted as crosses) stay within the region enclosed by the coordinate axes and the two quadrics as shown in Figure 8.1 (b).

In this section we want to show that the above picture in principle is the same for all scSPP, i.e., feasible, clean, and strongly connected SPPs. That is, we show that the Newton (and Kleene) approximants always stay in the region enclosed by the coordinate axes and the quadrics. We characterize this region and study some of the properties of the quadrics restricted to this region. This eventually leads to another method for approximating  $\mu\mathbf{f}$ . We then compare this new method with Newton’s method, thereby obtaining a unification of both methods. We close the section by discussing some properties of this more general method. All missing proofs can be found in the appendix.

Let us start with the properties of the quadrics  $Q_i$ . We restrict our attention to the region  $[\mathbf{0}, \mu\mathbf{f}]$ . For this we set

$$M_i := Q_i \cap [\mathbf{0}, \mu\mathbf{f}] = \{\mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}] \mid q_i(\mathbf{x}) = 0\}.$$

We start by showing that for every  $\mathbf{x} \in M_i$  the gradient  $q'_i(\mathbf{x})$  in  $\mathbf{x}$  at  $M_i$  does not vanish. As  $q'_i(\mathbf{x})$  is perpendicular to the tangent plane in  $\mathbf{x}$  at  $M_i$ , this means that the normal of the tangent plane is determined by  $q'_i(\mathbf{x})$  (up to orientation). See Figure 8.2 for an example. This will later allow us to apply the implicit function theorem.

LEMMA 8.2. *For every quadric  $q_i$  induced by a scSPP  $\mathbf{f}$  we have*

$$q'_i(\mathbf{x}) = (\partial_{X_1} q_i(\mathbf{x}), \partial_{X_2} q_i(\mathbf{x}), \dots, \partial_{X_n} q_i(\mathbf{x})) \neq \mathbf{0} \text{ and } \partial_{X_i} q_i(\mathbf{x}) < 0 \quad \forall \mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}].$$

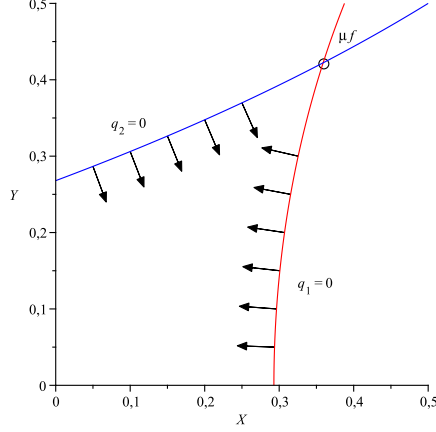


FIG. 8.2. The normals (scaled down) of the quadrics from Example 8.1.

In the following, for  $i \in \{1, \dots, n\}$  we write  $\mathbf{x}_{-i}$  for the vector  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  and define  $(\mathbf{x}_{-i}, x_i)$  to also denote the original vector  $\mathbf{x}$ .

We next show that there exists a complete parametrization of “the lower part” of  $M_i$ . With “lower part” we refer to the set

$$S_i := \{\mathbf{x} \in M_i \mid \forall \mathbf{y} \in M_i : (\mathbf{x}_{-i} = \mathbf{y}_{-i}) \Rightarrow x_i \leq y_i\}.$$

Taking a look at Figure 8.1, the surfaces  $S_1$  and  $S_2$  are those parts of  $M_1$ , resp.  $M_2$ , which delimit that part of  $\mathbb{R}_{\geq 0}^2$  shown in Figure 8.1 (b).

If  $\mathbf{x} \in S_i$  then  $x_i$  is the least non-negative root of the (at most) quadratic polynomial  $q_i(X_i, \mathbf{x}_{-i})$ . As we will see, these roots can also be represented by the following functions:

DEFINITION 8.3. For an scSPP  $\mathbf{f}$  we define for all  $k \in \mathbb{N}$  the polynomial  $h_i^{(k)}$  by

$$h_i^{(0)}(\mathbf{X}_{-i}) := f_i[i/0](\mathbf{X}_{-i}), \quad h_i^{(k+1)}(\mathbf{X}_{-i}) := f_i[i/h_i^{(k)}](\mathbf{X}_{-i})$$

The function  $h_i(\mathbf{X})$  is then defined pointwise by

$$h_i(\mathbf{x}_{-i}) := \lim_{k \rightarrow \infty} h_i^{(k)}(\mathbf{x}_{-i})$$

for all  $\mathbf{x}_{-i} \in [0, \mu \mathbf{f}_{-i}]$ .

We show in the appendix (see Proposition B.1) that the function  $h_i$  is well-defined and exists. We therefore can parameterize the surface  $S_i$  w.r.t. to the remaining variables  $\mathbf{X}_{-i}$ , i.e.,  $h_i$  is the “height” of the surface  $S_i$  above the “ground”  $X_i = 0$ .

By the preceding proposition the map

$$\mathbf{p}_i : [0, \mu \mathbf{f}_{-i}] \rightarrow [0, \mu \mathbf{f}] : \mathbf{x}_{-i} \mapsto (x_1, \dots, x_{i-1}, h_i(\mathbf{x}_{-i}), x_{i+1}, \dots, x_n)$$

gives us a pointwise parametrization of  $S_i$ . We want to show that  $\mathbf{p}_i$  is continuously differentiable. For this it suffices to show that  $h_i$  is continuously differentiable which follows easily from the implicit function theorem (see e.g. [27]).

LEMMA 8.4.  $h_i$  is continuously differentiable with

$$\partial_{X_j} h_i(\mathbf{x}_{-i}) = \frac{\partial_{X_j} f_i(\mathbf{x})}{-\partial_{X_i} q_i(\mathbf{x})} = \frac{\partial_{X_j} q_i(\mathbf{x})}{-\partial_{X_i} q_i(\mathbf{x})} \text{ for } \mathbf{x} \in S_i \text{ and } j \neq i.$$

In particular,  $\partial_{x_j} h_i$  is monotonically increasing with  $\mathbf{x}$ .

COROLLARY 8.5. The map

$$\mathbf{p}_i : [0, \mu \mathbf{f}_{-i}] \rightarrow [0, \mu \mathbf{f}] : \mathbf{x}_{-i} \mapsto (x_1, \dots, x_{i-1}, h_i(\mathbf{x}_{-i}), x_{i+1}, \dots, x_n)$$

is continuously differentiable and a local parametrization of the manifold  $S_i$ .

EXAMPLE 8.6. For the SPP  $\mathbf{f}$  defined in Example 8.1 we can simply solve  $q_1(X, Y)$  for  $X$  leading to

$$h_1(Y) = 1 - \sqrt{\frac{1}{2}(1 - Y^2)}.$$

The important point is that by the previous result we know that this function has to be defined on  $[0, \mu \mathbf{f}_2]$ , and differentiable on  $[0, \mu \mathbf{f}_2)$ . Similarly, we get

$$h_2(X) = 2 - \frac{1}{2}X - \frac{1}{2}\sqrt{X^2 - 12X + 12}.$$

Figure 8.1 (b) conveys the impression that the surfaces  $S_i$  are convex w.r.t. to the parameterizations  $\mathbf{p}_i$ . As we have seen, the functions  $h_i$  are monotonically increasing. Thus, in the case of two dimensions the functions  $h_i$  even have to be strictly monotonically increasing (as  $\mathbf{f}$  is strongly-connected), so that the surfaces  $S_i$  are indeed convex. (Recall that a surface  $S$  is convex in a point  $\mathbf{x} \in S$  if  $S$  is located completely on one side of the tangent plane at  $S$  in  $\mathbf{x}$ .) But in the case of more than two variables this does not need to hold anymore.

EXAMPLE 8.7. The equation

$$Z = \frac{1}{8}X^2 + \frac{3}{4}XY + \frac{1}{8}Y^2 + \frac{1}{4}$$

is an admissible part of any SPP. It defines the hyperbolic paraboloid depicted in Figure 8.3 which is clearly not convex.

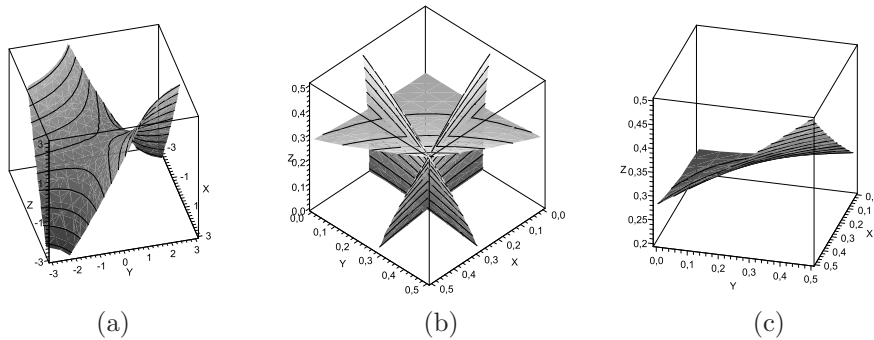


FIG. 8.3. (a) The hyperbolic paraboloid defined by  $Z = \frac{1}{8}X^2 + \frac{3}{4}XY + \frac{1}{8}Y^2 + \frac{1}{4}$  for  $X, Y, Z \in [-10, 10]$ . (b) A visualization of an SPP consisting of three copies of the quadric of (a) with  $\mu \mathbf{f} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  the upper apex. (c) One of the three quadrics of (b) over  $[0, \mu \mathbf{f}]$ . Clearly, even limited to this range the surface is not convex.

Still, as shown in Lemma 2.3 it holds for all  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$  that

$$\mathbf{x} + \mathbf{f}'(\mathbf{x}) \cdot \mathbf{y} \leq \mathbf{f}(\mathbf{x} + \mathbf{y}).$$

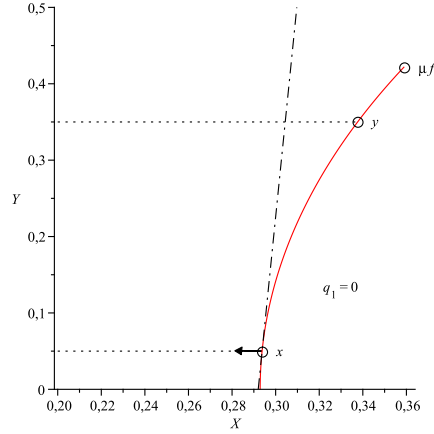


FIG. 8.4. The graphic shows the quadric defined by  $q_1 = 0$  with the tangent and normal in  $\mathbf{x}$  at  $S_1$ . Every point  $\mathbf{y}$  of  $S_1$  above  $\mathbf{x}$  is located on the same side of the the tangent. More precisely, we have  $\nabla q_1|_{\mathbf{x}} \cdot (\mathbf{y} - \mathbf{x}) \leq 0$ .

It now follows (see the following lemma) that the surfaces  $S_i$  have the property that for every  $\mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}]$  the “relevant” part of  $S_i$  for determining  $\mu\mathbf{f}$ , i.e.  $S_i \cap [\mathbf{x}, \mu\mathbf{f}]$ , is located on the same side of the tangent plane at  $S_i$  in  $\mathbf{x}$  (see Figure 8.4).

LEMMA 8.8. For all  $\mathbf{x} \in S_i$  we have

$$\forall \mathbf{y} \in S_i \cap [\mathbf{x}, \mu\mathbf{f}] : q'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \leq 0.$$

In particular

$$\forall \mathbf{y} \in S_i \cap [\mathbf{x}, \mu\mathbf{f}] : y_i \geq x_i + \sum_{j \neq i} \partial_{X_j} h_i(\mathbf{x}_{-i}) \cdot (y_j - x_j).$$

Consider now the set

$$R := \bigcap_{i=1}^n \{\mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}] \mid x_i \leq h_i(\mathbf{x}_{-i})\},$$

i.e., the region of  $[\mathbf{0}, \mu\mathbf{f}]$  delimited by the coordinate axes and the surfaces  $S_i$ . Note that the gradient  $q'_i(\mathbf{x})$  for  $\mathbf{x} \in S_i$  points from  $S_i$  into  $R$  (see Figure 8.2).

PROPOSITION 8.9. It holds

$$\mathbf{x} \in R \Leftrightarrow \mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}] \wedge \mathbf{q}(\mathbf{x}) \geq \mathbf{0}.$$

From this last result it now easily follows that  $R$  is indeed the region of  $[\mathbf{0}, \mu\mathbf{f}]$  where all Newton and Kleene steps are located in.

THEOREM 8.10. All Newton and Kleene steps starting from  $\mathbf{0}$  lie within  $R$ , i.e.

$$\boldsymbol{\nu}^{(i)}, \boldsymbol{\kappa}^{(i)} \in R \quad (\forall i \in \mathbb{N}).$$

*Proof.* For an scSPP we have  $\boldsymbol{\kappa}^{(i)}, \boldsymbol{\nu}^{(i)} \in [\mathbf{0}, \mu\mathbf{f}]$  for all  $i$ . Further,  $\boldsymbol{\kappa}^{(i)} \leq \boldsymbol{\kappa}^{(i+1)} = \mathbf{f}(\boldsymbol{\kappa}^{(i)})$  and  $\boldsymbol{\nu}^{(i)} \leq \mathbf{f}(\boldsymbol{\nu}^{(i)})$  holds for all  $i$ , too.  $\square$

In the rest of this section we will use the results regarding  $R$  and the surfaces  $S_i$  for interpreting Newton's method geometrically and for obtaining a generalization of Newton's method:

The preceding results suggest another way of determining  $\mu\mathbf{f}$  (see Figure 8.5): Let  $\mathbf{x}$  be some point inside of  $R$ . We may move from  $\mathbf{x}$  onto one of the surface  $S_i$  by going upward along the line  $\mathbf{x} + t \cdot \mathbf{e}_i$  which gives us the point  $\mathbf{p}_i(\mathbf{x}_{-i}) = (\mathbf{x}_{-i}, h_i(\mathbf{x}_{-i}))$ . As  $\mathbf{x} \in R$ , we have  $\mathbf{x}, \mathbf{p}_i(\mathbf{x}_{-i}) \leq \mu\mathbf{f}$ . Consider now the tangent plane

$$T_i|_{\mathbf{x}} = \{\mathbf{y} \in \mathbb{R}^n \mid q'_i(\mathbf{p}_i(\mathbf{x}_{-i})) \cdot (\mathbf{y} - \mathbf{p}_i(\mathbf{x}_{-i})) = 0\}$$

at  $S_i$  in  $\mathbf{p}_i(\mathbf{x}_{-i})$ . Recall that by Lemma 8.8 we have

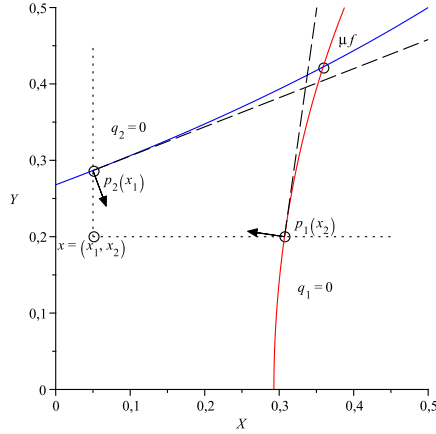


FIG. 8.5. Given a point  $\mathbf{x}$  inside of  $R$  the intersection of the tangents at the quadrics in the points  $\mathbf{p}_1(\mathbf{x}_2)$ , resp.  $\mathbf{p}_2(\mathbf{x}_1)$  is also located inside of  $R$ , yielding a better approximation of  $\mu\mathbf{f}$ .

$$\forall \mathbf{y} \in S_i \cap [\mathbf{p}_i(\mathbf{x}_{-i}), \mu\mathbf{f}] : q'_i(\mathbf{p}_i(\mathbf{x}_{-i})) \cdot (\mathbf{y} - \mathbf{p}_i(\mathbf{x}_{-i})) \leq 0,$$

i.e., the part of  $S_i$  relevant for determining  $\mu\mathbf{f}$  is located completely below (w.r.t.  $q'_i(\mathbf{p}_i(\mathbf{x}_{-i}))$ ) this tangent plane. By continuity this also has to hold for  $\mathbf{y} = \mu\mathbf{f}$ . Hence, when taking the intersection of all the tangent planes  $T_1$  to  $T_n$  this gives us again a point  $\mathcal{T}(\mathbf{x})$  inside of  $R$ . That this point  $\mathcal{T}(\mathbf{x})$  exists and is uniquely determined is shown in the following lemma.

LEMMA 8.11. Let  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in [\mathbf{0}, \mu\mathbf{f}]$ . Then the matrix

$$\begin{pmatrix} q'_1(\mathbf{x}^{(1)}) \\ \vdots \\ q'_n(\mathbf{x}^{(n)}) \end{pmatrix}$$

is regular, i.e., the vectors  $\{q'_i(\mathbf{x}^{(i)}) \mid i = 1, \dots, n\}$  are linearly independent.

By this lemma the normals at the quadrics in the points  $\mathbf{p}_i(\mathbf{x}_{-i})$  for  $\mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}]$  are linearly independent. Thus, there exists a unique point of intersection of tangent planes at the quadrics in these points.

DEFINITION 8.12. For  $\mathbf{x} \in R$  define  $\mathcal{T}(\mathbf{x})$  to be the unique solution of the equation

system

$$\begin{aligned} q'_1(\mathbf{p}_1(\mathbf{x}_{-1})) \cdot \mathbf{X} &= q'_1(\mathbf{p}_1(\mathbf{x}_{-1})) \cdot \mathbf{p}_1(\mathbf{x}_{-1}) \\ &\vdots \\ q'_n(\mathbf{p}_n(\mathbf{x}_{-n})) \cdot \mathbf{X} &= q'_n(\mathbf{p}_n(\mathbf{x}_{-n})) \cdot \mathbf{p}_n(\mathbf{x}_{-n}) \end{aligned}$$

THEOREM 8.13. *We have*

$$\mathbf{x} \leq \mathcal{T}(\mathbf{x}) \text{ and } \mathcal{N}(\mathbf{x}) \leq \mathcal{T}(\mathbf{x}) \leq \mu \mathbf{f}.$$

Further, for  $\mathbf{y} \in R$  with  $\mathbf{x} \leq \mathbf{y}$

$$\mathcal{T}(\mathbf{x}) \leq \mathcal{T}(\mathbf{y}).$$

See Figure 8.6 for a geometrical interpretation of Newton's method.

**9. Conclusions.** We have studied the convergence order and convergence rate of Newton's method for fixed-point equations of systems of positive polynomials (SPP equations). These equations appear naturally in the analysis of several stochastic computational models that have been intensely studied in recent years, and they also play a central rôle in the theory of stochastic branching processes.

The restriction to positive coefficients leads to strong results. For arbitrary polynomial equations Newton's method may not converge or converge only locally, i.e., when started at a point sufficiently close to the solution. We have extended a result by Etessami and Yannakakis [14], and shown that for SPP equations the method always converges starting at  $\mathbf{0}$ . Moreover, we have proved that the method has at least linear convergence order, and have determined the asymptotic convergence rate. To the best of our knowledge, this is the first time that a lower bound on the convergence order is proved for a significant class of equations with a trivial membership test.<sup>5</sup> Finally, we have also obtained upper bounds on the threshold, i.e., the number of iterations necessary to reach the "steady state" in which valid bits are computed at the asymptotic rate. These results lead to practical tests for checking whether the least fixed point of an SPP exceeds a given bound.

There are still at least two important open questions. The behavior of Newton's method when arithmetic operations only have a fixed accuracy should be further investigated. We wish to develop tests allowing to decide whether the result of applying Newton's method with a certain fixed accuracy is reliable or not. For the second question, say that Newton's method is *polynomial* for a class of SPP equations if there is a polynomial  $p(x, y, z)$  such that for every  $k \geq 0$  and for every system in the class with  $n$  equations and coefficients of size  $m$ , the  $p(n, m, k)$ -th Newton approximant  $\nu^{(p(n, m, k))}$  has  $k$  valid bits. We have proved in Theorem 5.12 that Newton's method is polynomial for SPPs  $\mathbf{f}$  satisfying  $\mathbf{f}(\mathbf{0}) \succ \mathbf{0}$ ; for this class one can take  $p(n, m, k) = 7mn + k$ . We have also exhibited in § 7 a class for which computing the first bit of the least solution takes  $2^n$  iterations. The members of this class, however, are non-strongly-connected, and this is the fact we have exploited to construct them. So the following question remains open: Is Newton's method polynomial for strongly connected SPPs?

### Appendix A. Proof of Lemma 6.3.

<sup>5</sup>Notice the contrast with the classical result stating that if  $(\text{Id} - \mathbf{f}'(\mu \mathbf{f}))$  is non-singular, then Newton's method has exponential convergence order; here the membership test is highly non-trivial, and, for what we know, as hard as computing  $\mu \mathbf{f}$  itself).



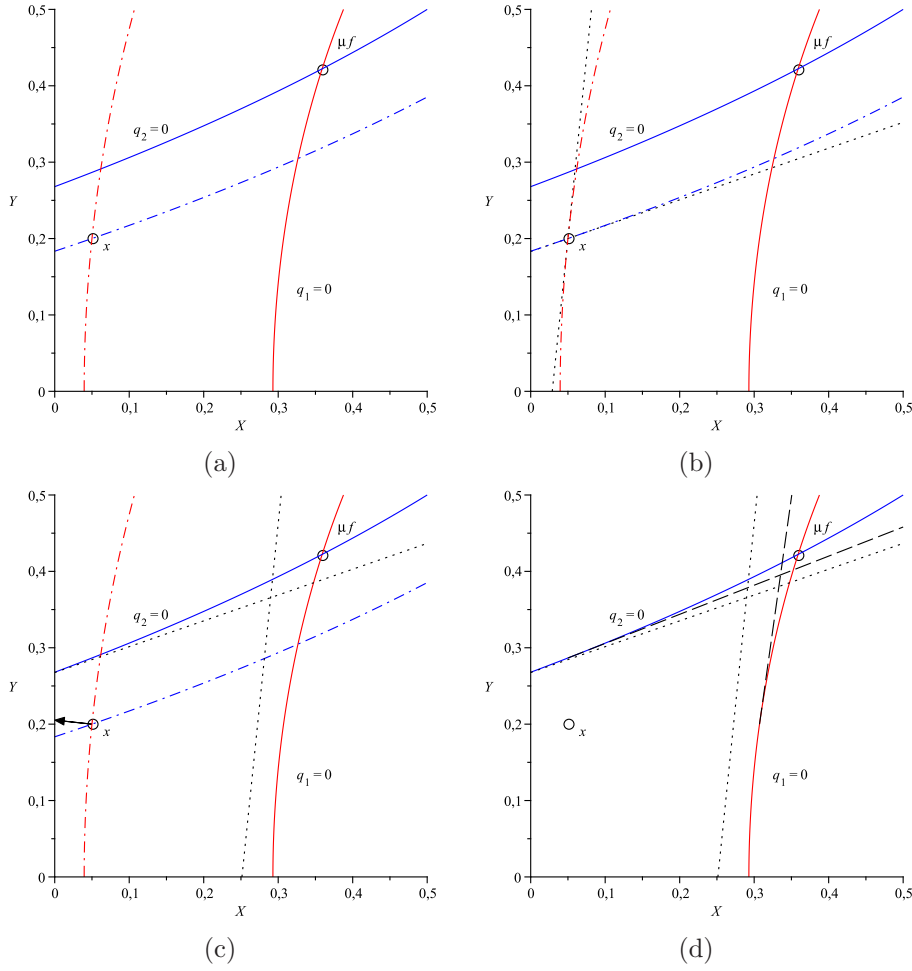


FIG. 8.6. Geometrical interpretation of Newton's method: (a) Given a point  $\mathbf{x} \in R$  Newton's method first considers the "enlarged" quadrics defined by  $q_i(\mathbf{X}) = q_i(\mathbf{x})$  (drawn dashed and dotted) which contain the current approximation  $\mathbf{x}$ . (b) Then the tangents in  $\mathbf{x}$  at these enlarged quadrics are computed (drawn dotted), i.e.,  $q'_i(\mathbf{x}) \cdot (\mathbf{X} - \mathbf{x}) = 0$ . (c) Finally, these tangents are corrected by moving them towards the actual quadrics, i.e.  $q'_i(\mathbf{x}) \cdot (\mathbf{X} - \mathbf{x}) = -q_i(\mathbf{x})$ . The intersection of these corrected tangents gives the next Newton approximation. (d) A comparison between  $N(\mathbf{x})$  and  $T(\mathbf{x})$ :  $N(\mathbf{x})$ , resp.  $T(\mathbf{x})$  is given by the intersection of the dotted, resp. dashed lines. Clearly, we have  $N(\mathbf{x}) \leq T(\mathbf{x})$ .

The proof of Lemma 6.3 is by a sequence of lemmata. The following two Lemmata A.1 and A.2 provide a lower bound on  $\|\mathbf{f}(\mathbf{x}) - \mathbf{x}\|$  for an "almost-fixed-point"  $\mathbf{x}$ .

LEMMA A.1. Let  $\mathbf{f}$  be a quadratic SPP without linear terms, i.e.,  $\mathbf{f}(\mathbf{X}) = B(\mathbf{X}, \mathbf{X}) + \mathbf{c}$  where  $B$  is a bilinear map, and  $\mathbf{c}$  is a constant vector. Let  $\mathbf{f}(\mathbf{X})$  be non-constant in every component. Let  $R \dot{\cup} S = \{1, \dots, n\}$  with  $S \neq \emptyset$ . Let every component depend on every  $S$ -component and not on any  $R$ -component. Then there is a constant  $C_f > 0$  such that

$$\|\mathbf{f}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta)\| \geq C_f \cdot \|\delta\|^2$$

for all  $\delta$  with  $\mathbf{0} \leq \delta \leq \mu\mathbf{f}$ .

*Proof.* With the given component dependencies we can write  $\mathbf{f}(\mathbf{X})$  as follows:

$$\mathbf{f}_R(\mathbf{X}) = \begin{pmatrix} \mathbf{f}_R(\mathbf{X}) \\ \mathbf{f}_S(\mathbf{X}) \end{pmatrix} = \begin{pmatrix} B_R(\mathbf{X}_S, \mathbf{X}_S) + \mathbf{c}_R \\ B_S(\mathbf{X}_S, \mathbf{X}_S) + \mathbf{c}_S \end{pmatrix}$$

A straightforward calculation shows

$$\mathbf{e}(\boldsymbol{\delta}) := \mathbf{f}(\boldsymbol{\mu}\mathbf{f} - \boldsymbol{\delta}) - (\boldsymbol{\mu}\mathbf{f} - \boldsymbol{\delta}) = (\text{Id} - \mathbf{f}'(\boldsymbol{\mu}\mathbf{f}))\boldsymbol{\delta} + B(\boldsymbol{\delta}, \boldsymbol{\delta}).$$

Furthermore,  $\partial_{\mathbf{X}_R}\mathbf{f}$  is constant zero in all entries, so

$$\begin{aligned} e_R(\boldsymbol{\delta}) &= \boldsymbol{\delta}_R - \partial_{\mathbf{X}_S}\mathbf{f}_R(\boldsymbol{\mu}\mathbf{f}) \cdot \boldsymbol{\delta}_S + B_R(\boldsymbol{\delta}_S, \boldsymbol{\delta}_S) \quad \text{and} \\ e_S(\boldsymbol{\delta}) &= \boldsymbol{\delta}_S - \partial_{\mathbf{X}_S}\mathbf{f}_S(\boldsymbol{\mu}\mathbf{f}) \cdot \boldsymbol{\delta}_S + B_S(\boldsymbol{\delta}_S, \boldsymbol{\delta}_S). \end{aligned}$$

Notice that for every real number  $r > 0$  we have

$$\min_{\mathbf{0} \leq \boldsymbol{\delta} \leq \boldsymbol{\mu}\mathbf{f}, \|\boldsymbol{\delta}\| \geq r} \frac{\|\mathbf{e}(\boldsymbol{\delta})\|^2}{\|\boldsymbol{\delta}\|} > 0,$$

because otherwise  $\boldsymbol{\mu}\mathbf{f} - \boldsymbol{\delta} < \boldsymbol{\mu}\mathbf{f}$  would be a fixed point of  $\mathbf{f}$ . We have to show:

$$\inf_{\mathbf{0} \leq \boldsymbol{\delta} \leq \boldsymbol{\mu}\mathbf{f}, \|\boldsymbol{\delta}\| > 0} \frac{\|\mathbf{e}(\boldsymbol{\delta})\|^2}{\|\boldsymbol{\delta}\|} > 0$$

Assume, for a contradiction, that this infimum equals zero. Then there exists a sequence  $(\boldsymbol{\delta}^{(i)})_{i \in \mathbb{N}}$  with  $\mathbf{0} \leq \boldsymbol{\delta}^{(i)} \leq \boldsymbol{\mu}\mathbf{f}$ ,  $\|\boldsymbol{\delta}^{(i)}\| > 0$  such that  $\lim_{i \rightarrow \infty} \|\boldsymbol{\delta}^{(i)}\| = 0$  and  $\lim_{i \rightarrow \infty} \frac{\|\mathbf{e}(\boldsymbol{\delta}^{(i)})\|}{\|\boldsymbol{\delta}^{(i)}\|^2} = 0$ . Define  $r^{(i)} := \|\boldsymbol{\delta}^{(i)}\|$  and  $\mathbf{d}^{(i)} := \frac{\boldsymbol{\delta}^{(i)}}{\|\boldsymbol{\delta}^{(i)}\|}$ . Notice that  $\mathbf{d}^{(i)} \in \{\mathbf{d} \in \mathbb{R}_{\geq 0}^n \mid \|\mathbf{d}\| = 1\} =: D$  where  $D$  is compact. So some subsequence of  $(\mathbf{d}^{(i)})_{i \in \mathbb{N}}$ , say w.l.o.g. the sequence  $(\mathbf{d}^{(i)})_{i \in \mathbb{N}}$  itself, converges to some vector  $\mathbf{d}^*$ . By our assumption we have

$$\mathbf{e}(\boldsymbol{\delta}^{(i)}) / \|\boldsymbol{\delta}^{(i)}\|^2 = \left\| \frac{1}{r^{(i)}} (\text{Id} - \mathbf{f}'(\boldsymbol{\mu}\mathbf{f}))\mathbf{d}^{(i)} + B(\mathbf{d}^{(i)}, \mathbf{d}^{(i)}) \right\| \longrightarrow 0. \quad (\text{A.1})$$

As  $B(\mathbf{d}^{(i)}, \mathbf{d}^{(i)})$  is bounded,  $\frac{1}{r^{(i)}}(\text{Id} - \mathbf{f}'(\boldsymbol{\mu}\mathbf{f}))\mathbf{d}^{(i)}$  must be bounded, too. Since  $r^{(i)}$  converges to 0,  $\left\| (\text{Id} - \mathbf{f}'(\boldsymbol{\mu}\mathbf{f}))\mathbf{d}^{(i)} \right\|$  must converge to 0, so

$$(\text{Id} - \mathbf{f}'(\boldsymbol{\mu}\mathbf{f}))\mathbf{d}^* = \mathbf{0}.$$

In particular,  $((\text{Id} - \mathbf{f}'(\boldsymbol{\mu}\mathbf{f}))\mathbf{d}^*)_R = \mathbf{d}_R^* - \partial_{\mathbf{X}_S}\mathbf{f}_R(\boldsymbol{\mu}\mathbf{f}) \cdot \mathbf{d}_S^* = \mathbf{0}$ . So we have  $\mathbf{d}_S^* > \mathbf{0}$ , because  $\mathbf{d}_S^* = \mathbf{0}$  would imply  $\mathbf{d}_R^* = \mathbf{0}$  which would contradict  $\mathbf{d}^* > \mathbf{0}$ .

In the remainder of the proof we focus on  $\mathbf{f}_S$ . Define the scSPP  $\mathbf{g}(\mathbf{X}_S) := \mathbf{f}_S(\mathbf{X})$ . Notice that  $\boldsymbol{\mu}\mathbf{g} = \boldsymbol{\mu}\mathbf{f}_S$ . We can apply Lemma 5.3 to  $\mathbf{g}$  and  $\mathbf{d}_S^*$  and obtain  $\mathbf{d}_S^* \succ \mathbf{0}$ . As  $\mathbf{f}_S(\mathbf{X})$  is non-constant we get  $B_S(\mathbf{d}_S^*, \mathbf{d}_S^*) \succ \mathbf{0}$ . By (A.1),  $\frac{1}{r^{(i)}}(\text{Id} - \mathbf{g}'(\boldsymbol{\mu}\mathbf{g}))\mathbf{d}_S^{(i)}$  converges to  $-B_S(\mathbf{d}_S^*, \mathbf{d}_S^*) \prec \mathbf{0}$ . So there is a  $j \in \mathbb{N}$  such that  $(\text{Id} - \mathbf{g}'(\boldsymbol{\mu}\mathbf{g}))\mathbf{d}_S^{(j)} \prec \mathbf{0}$ . Let  $\tilde{\boldsymbol{\delta}} := r\mathbf{d}^{(j)}$  for some small enough  $r > 0$  such that  $\mathbf{0} < \tilde{\boldsymbol{\delta}}^* \leq \boldsymbol{\mu}\mathbf{g}$  and

$$\begin{aligned} e_S(\tilde{\boldsymbol{\delta}}) &= (\text{Id} - \mathbf{g}'(\boldsymbol{\mu}\mathbf{g}))\tilde{\boldsymbol{\delta}}_S + B_S(\tilde{\boldsymbol{\delta}}_S, \tilde{\boldsymbol{\delta}}_S) \\ &= r(\text{Id} - \mathbf{g}'(\boldsymbol{\mu}\mathbf{g}))\mathbf{d}_S^{(j)} + r^2 B_S(\mathbf{d}_S^{(j)}, \mathbf{d}_S^{(j)}) \prec \mathbf{0}. \end{aligned}$$

So we have  $\mathbf{g}(\mu\mathbf{g} - \tilde{\delta}_S) \prec \mu\mathbf{g} - \tilde{\delta}_S$ . However,  $\mu\mathbf{g}$  is the least point  $\mathbf{x}$  with  $\mathbf{g}(\mathbf{x}) \leq \mathbf{x}$ . Thus we get the desired contradiction.  $\square$

LEMMA A.2. *Let  $\mathbf{f}$  be a quadratic strongly connected SPP. Then there is a constant  $C_{\mathbf{f}} > 0$  such that*

$$\|\mathbf{f}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta)\| \geq C_{\mathbf{f}} \cdot \|\delta\|^2$$

for all  $\delta$  with  $\mathbf{0} \leq \delta \leq \mu\mathbf{f}$ .

*Proof.* Write  $\mathbf{f}(\mathbf{X}) = B(\mathbf{X}, \mathbf{X}) + L\mathbf{X} + \mathbf{c}$  for a bilinear map  $B$ , a matrix  $L$  and a constant vector  $\mathbf{c}$ . By Theorem 4.8.2. the matrix  $L^* = (\text{Id} - L)^{-1} = (\text{Id} - \mathbf{f}'(\mathbf{0}))^{-1}$  exists. Define the SPP  $\tilde{\mathbf{f}}(\mathbf{X}) := L^*B(\mathbf{X}, \mathbf{X}) + L^*\mathbf{c}$ . A straightforward calculation shows that the sets of fixed points of  $\mathbf{f}$  and  $\tilde{\mathbf{f}}$  coincide and that

$$\mathbf{f}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta) = (\text{Id} - L) \left( \tilde{\mathbf{f}}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta) \right).$$

Further we have

$$\left\| (\text{Id} - L) \left( \tilde{\mathbf{f}}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta) \right) \right\|_2 \geq \sigma_1(\text{Id} - L) \left\| \tilde{\mathbf{f}}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta) \right\|_2$$

where  $\sigma_1(\text{Id} - L)$  denotes the smallest singular value of  $\text{Id} - L$ . Note that  $\sigma_1(\text{Id} - L) > 0$  because  $\text{Id} - L$  is invertible. So it suffices to show that

$$\left\| \tilde{\mathbf{f}}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta) \right\| \geq C_{\mathbf{f}} \cdot \|\delta\|^2.$$

If  $\mathbf{f}(\mathbf{X})$  is linear (i.e.  $B(\mathbf{X}, \mathbf{X}) \equiv \mathbf{0}$ ) then  $\tilde{\mathbf{f}}(\mathbf{X})$  is constant and we have  $\left\| \tilde{\mathbf{f}}(\mu\mathbf{f} - \delta) - (\mu\mathbf{f} - \delta) \right\| = \|\delta\|$ , so we are done in that case. Hence we can assume that some component of  $B(\mathbf{X}, \mathbf{X})$  is not the zero polynomial. It remains to argue that  $\tilde{\mathbf{f}}$  satisfies the preconditions of Lemma A.1. By definition,  $\tilde{\mathbf{f}}$  does not have linear terms. Define

$$S := \{i \mid 1 \leq i \leq n, X_i \text{ is contained in a component of } B(\mathbf{X}, \mathbf{X})\}.$$

Notice that  $S$  is non-empty. Let  $i_0, i_1, \dots, i_m, i_{m+1}$  ( $m \geq 0$ ) be any sequence such that, in  $\mathbf{f}$ , for all  $j$  with  $0 \leq j < m$  the component  $i_j$  depends directly on  $i_{j+1}$  via a linear term and  $i_m$  depends directly on  $i_{m+1}$  via a quadratic term. Then  $i_0$  depends directly on  $i_{m+1}$  via a quadratic term in  $L^m B(\mathbf{X}, \mathbf{X})$  and hence also in  $\tilde{\mathbf{f}}$ . So all components are non-constant and depend (directly or indirectly) on every  $S$ -component. Furthermore, no component depends on a component that is not in  $S$ , because  $L^*B(\mathbf{X}, \mathbf{X})$  contains only  $S$ -components. Thus, Lemma A.1 can be applied, and the statement follows.  $\square$

The following lemma gives a bound on the propagation error for the case that  $\mathbf{f}$  has a single top SCC.

LEMMA A.3. *Let  $\mathbf{f}$  be a quadratic SPP. Let  $S \subseteq \{1, \dots, n\}$  be the single top SCC of  $\mathbf{f}$ . Let  $L := \{1, \dots, n\} \setminus S$ . Then there is a constant  $C_{\mathbf{f}} \geq 0$  such that*

$$\|\mu\mathbf{f}_S - \tilde{\mu}_S\| \leq C_{\mathbf{f}} \cdot \sqrt{\|\mu\mathbf{f}_L - \mathbf{x}_L\|}$$

for all  $\mathbf{x}_L$  with  $\mathbf{0} \leq \mathbf{x}_L \leq \mu\mathbf{f}_L$  where  $\tilde{\mu}_S := \mu(\mathbf{f}_S[\mathbf{X}_L/\mathbf{x}_L])$ .

*Proof.* We write  $\mathbf{f}_S(\mathbf{X}) = \mathbf{f}_S(\mathbf{X}_S, \mathbf{X}_L)$  in the following.

If  $S$  is a trivial SCC then  $\mu\mathbf{f}_S = \mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L)$  and  $\tilde{\mu}_S = \mathbf{f}_S(\mathbf{0}, \mathbf{x}_L)$ . In this case we have with Taylor's theorem (cf. Lemma 2.3)

$$\begin{aligned} \|\mu\mathbf{f}_S - \tilde{\mu}_S\| &= \|\mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L) - \mathbf{f}_S(\mathbf{0}, \mathbf{x}_L)\| \\ &\leq \|\partial_{\mathbf{X}}\mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L) \cdot (\mu\mathbf{f}_L - \mathbf{x}_L)\| \\ &\leq \|\partial_{\mathbf{X}}\mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L)\| \cdot \|\mu\mathbf{f}_L - \mathbf{x}_L\| \\ &= \|\partial_{\mathbf{X}}\mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L)\| \cdot \sqrt{\|\mu\mathbf{f}_L - \mathbf{x}_L\|} \cdot \sqrt{\|\mu\mathbf{f}_L - \mathbf{x}_L\|} \\ &\leq \|\partial_{\mathbf{X}}\mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L)\| \cdot \sqrt{\|\mu\mathbf{f}_L\|} \cdot \sqrt{\|\mu\mathbf{f}_L - \mathbf{x}_L\|} \end{aligned}$$

and the statement follows by setting  $C_f := \|\partial_{\mathbf{X}}\mathbf{f}_S(\mathbf{0}, \mu\mathbf{f}_L)\| \cdot \sqrt{\|\mu\mathbf{f}_L\|}$ .

Hence, in the following we can assume that  $S$  is a non-trivial SCC. Set  $\mathbf{g}(\mathbf{X}_S) := \mathbf{f}_S(\mathbf{X}_S, \mu\mathbf{f}_L)$ . Notice that  $\mathbf{g}$  is an scSPP with  $\mu\mathbf{g} = \mu\mathbf{f}_S$ . By applying Lemma A.2 to  $\mathbf{g}$  and setting  $c := 1/\sqrt{C_g}$  (the  $C_g$  from Lemma A.2) we get:

$$\begin{aligned} \|\mu\mathbf{f}_S - \tilde{\mu}_S\| &\leq c \cdot \sqrt{\|\mathbf{g}(\mu\mathbf{g} - (\mu\mathbf{f}_S - \tilde{\mu}_S)) - (\mu\mathbf{g} - (\mu\mathbf{f}_S - \tilde{\mu}_S))\|} \\ &= c \cdot \sqrt{\|\mathbf{f}_S(\tilde{\mu}_S, \mu\mathbf{f}_L) - \tilde{\mu}_S\|} \\ &= c \cdot \sqrt{\|\mathbf{f}_S(\tilde{\mu}_S, \mu\mathbf{f}_L) - \mathbf{f}_S(\tilde{\mu}_S, \mathbf{x}_L)\|} \end{aligned}$$

with Taylor's theorem (cf. Lemma 2.3):

$$\begin{aligned} &\leq c \cdot \sqrt{\|\partial_{\mathbf{X}_L}\mathbf{f}_S(\tilde{\mu}_S, \mu\mathbf{f}_L)(\mu\mathbf{f}_L - \mathbf{x}_L)\|} \\ &\leq c \cdot \sqrt{\|\partial_{\mathbf{X}_L}\mathbf{f}_S(\mu\mathbf{f}_S, \mu\mathbf{f}_L)(\mu\mathbf{f}_L - \mathbf{x}_L)\|} \\ &\leq c \cdot \sqrt{\|\partial_{\mathbf{X}_L}\mathbf{f}_S(\mu\mathbf{f}_S, \mu\mathbf{f}_L)\|} \cdot \sqrt{\|\mu\mathbf{f}_L - \mathbf{x}_L\|} \end{aligned}$$

So the statement follows by setting  $C_f := c \cdot \sqrt{\|\partial_{\mathbf{X}_L}\mathbf{f}_S(\mu\mathbf{f}_S, \mu\mathbf{f}_L)\|}$ .  $\square$

Now we can extend Lemma A.3 to Lemma 6.3, restated here.

LEMMA 6.3. *There is a constant  $C_f > 0$  such that*

$$\|\mu\mathbf{f}_{[t]} - \tilde{\mu}_{[t]}\| \leq C_f \cdot \sqrt{\|\mu\mathbf{f}_{[>t]} - \rho_{[>t]}\|}$$

holds for all  $\rho_{[>t]}$  with  $\mathbf{0} \leq \rho_{[>t]} \leq \mu\mathbf{f}_{[>t]}$ , where  $\tilde{\mu}_{[t]} = \mu(\mathbf{f}_{[t]}[[>t]/\rho_{[>t]})$ .

*Proof.* Observe that  $\mu\mathbf{f}_{[t]}$ ,  $\tilde{\mu}_{[t]}$ ,  $\mu\mathbf{f}_{[>t]}$  and  $\rho_{[>t]}$  do not depend on the components of depth  $< t$ . So we can assume w.l.o.g. that  $t = 0$ . Let  $\text{SCC}(0) = \{S_1, \dots, S_k\}$ .

For any  $S_i$  from  $\text{SCC}(0)$ , let  $\mathbf{f}^{(i)}$  be obtained from  $\mathbf{f}$  by removing all top SCCs except for  $S_i$ . Lemma A.2 applied to  $\mathbf{f}^{(i)}$  guarantees a  $C^{(i)}$  such that

$$\|\mu\mathbf{f}_{S_i} - \tilde{\mu}_{S_i}\| \leq C^{(i)} \cdot \sqrt{\|\mu\mathbf{f}_{[>0]} - \rho_{[>0]}\|}$$

holds for all  $\rho_{[>0]}$  with  $\mathbf{0} \leq \rho_{[>0]} \leq \mu\mathbf{f}_{[>0]}$ . Using the equivalence of norms let w.l.o.g. the norm  $\|\cdot\|$  be the maximum-norm  $\|\cdot\|_{\infty}$ . Let  $C_f := \max_{1 \leq i \leq k} C^{(i)}$ . Then we have

$$\|\mu\mathbf{f}_{[0]} - \tilde{\mu}_{[0]}\| = \max_{1 \leq i \leq k} \|\mu\mathbf{f}_{S_i} - \tilde{\mu}_{S_i}\| \leq C_f \cdot \sqrt{\|\mu\mathbf{f}_{[>0]} - \rho_{[>0]}\|}$$

for all  $\rho_{[>0]}$  with  $\mathbf{0} \leq \rho_{[>0]} \leq \mu\mathbf{f}_{[>0]}$ .  $\square$

**Appendix B. Proofs of § 8.**

LEMMA 8.2. *For every quadric  $q_i$  induced by a scSPP  $\mathbf{f}$  we have*

$$q'_i(\mathbf{x}) = (\partial_{X_1} q_i(\mathbf{x}), \partial_{X_2} q_i(\mathbf{x}), \dots, \partial_{X_n} q_i(\mathbf{x})) \neq \mathbf{0} \text{ and } \partial_{X_i} q_i(\mathbf{x}) < 0 \quad \forall \mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}].$$

*Proof.* As shown by Etessami and Yannakakis in [14] under the above preconditions it holds for all  $\mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}]$  that  $(\text{Id} - \mathbf{f}'(\mathbf{x}))$  is invertible with

$$(\text{Id} - \mathbf{f}'(\mathbf{x}))^{-1} = \mathbf{f}'(\mathbf{x})^*.$$

Thus, we have

$$\mathbf{q}'(\mathbf{x})^{-1} = (\mathbf{f}'(\mathbf{x}) - \text{Id})^{-1} = -(\mathbf{f}'(\mathbf{x})^*),$$

implying that  $q'_i(\mathbf{x}) \neq 0$  for all  $\mathbf{x} \in [\mathbf{0}, \mu\mathbf{f}]$  as  $\mathbf{q}'(\mathbf{x})$  has to have full rank  $n$  in order for  $\mathbf{q}'(\mathbf{x})^{-1}$  to exist. Furthermore, it follows that all entries of  $\mathbf{q}'(\mathbf{x})^{-1}$  are non-positive as  $\mathbf{f}'(\mathbf{x})^*$  is non-negative. Now, as  $q_i(\mathbf{X}) = f_i(\mathbf{X}) - X_i$  and  $f_i(\mathbf{X})$  is a polynomial with non-negative coefficients, it holds that

$$q'_i(\mathbf{x}) \cdot \mathbf{e}_j = \partial_{X_j} q_i(\mathbf{x}) = \partial_{X_j} f_i(\mathbf{x}) \geq 0$$

for all  $j \neq i$  and  $\mathbf{x} \geq \mathbf{0}$ . With every entry of  $\mathbf{q}'(\mathbf{x})^{-1}$  non-positive, and

$$q'_i(\mathbf{x}) \cdot \mathbf{q}'(\mathbf{x})^{-1} = \mathbf{e}_i^\top,$$

we conclude  $\partial_{X_i} q_i(\mathbf{x}) < 0$ .  $\square$

PROPOSITION B.1. *Let  $\mathbf{x}, \mathbf{y} \in [\mathbf{0}, \mu\mathbf{f}]$  with  $\mathbf{x} \leq \mathbf{y}$ .*

(a)  $0 \leq h_i^{(k)}(\mathbf{x}_{-i}) \leq \mu\mathbf{f}_i$ .

(b)  $h_i^{(k)}(\mathbf{x}_{-i}) \leq h_i^{(k+1)}(\mathbf{x}_{-i})$  for all  $k \in \mathbb{N}$ .

(c)  $h_i^{(k)}(\mathbf{x}_{-i}) \leq h_i^{(k)}(\mathbf{y}_{-i})$  for all  $k \in \mathbb{N}$ .

(d)  $h_i(\mathbf{x}_{-i}) \leq \mu\mathbf{f}_i$ , and  $h_i$  is a map from  $[\mathbf{0}, \mu\mathbf{f}_{-i}]$  to  $[0, \mu\mathbf{f}_i]$ .

*If  $f_i$  depends on at least one other variable except  $X_i$ , we also have*

$$h_i([\mathbf{0}, \mu\mathbf{f}_{-i}]) \subseteq [0, \mu\mathbf{f}_i].$$

(e)  $h_i(\mathbf{x}_{-i}) \leq h_i(\mathbf{y}_{-i})$ .

(f)  $f_i(\mathbf{x}_{-i}, h_i(\mathbf{x}_{-i})) = h_i(\mathbf{x}_{-i})$ .

(g) For  $x_i = f_i(\mathbf{x})$  we have  $h_i(\mathbf{x}_{-i}) \leq x_i$ .

(h)  $h_i(\mu\mathbf{f}_{-i}) = \mu\mathbf{f}_i$ .

*Proof.* Let  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y} \leq \mu\mathbf{f}$ . Using the monotonicity of  $f_i$  over  $\mathbb{R}_{\geq 0}^n$  we proceed by induction on  $k$ .

(a) For  $k = 0$  we have

$$0 \leq h_i^{(0)}(\mathbf{x}_{-i}) = f_i(0, \mathbf{x}_{-i}) \leq f_i(\mu\mathbf{f}) = \mu\mathbf{f}_i.$$

We then get

$$0 \leq h_i^{(k+1)}(\mathbf{x}_{-i}) = f_i(h_i^{(k)}(\mathbf{x}_{-i}), \mathbf{x}_{-i}) \leq f_i(\mu\mathbf{f}) = \mu\mathbf{f}_i.$$

(b) For  $k = 0$  we have

$$h_i^{(0)}(\mathbf{x}_{-i}) = f_i(0, \mathbf{x}_{-i}) \leq f_i(h_i^{(0)}(\mathbf{x}_{-i}), \mathbf{x}_{-i}) = h_i^{(1)}(\mathbf{x}_{-i}).$$

Thus

$$h_i^{(k+1)}(\mathbf{x}_{-i}) = f_i(h_i^{(k)}(\mathbf{x}_{-i}), \mathbf{x}_{-i}) \leq f_i(h_i^{(k+1)}(\mathbf{x}_{-i}), \mathbf{x}_{-i}) = h_i^{(k+2)}(\mathbf{x}_{-i})$$

follows.

(c) As  $\mathbf{x} \leq \mathbf{y}$ , we have for  $k = 0$

$$h_i^{(0)}(\mathbf{x}_{-i}) = f_i(0, \mathbf{x}_{-i}) \leq f_i(0, \mathbf{y}_{-i}) = h_i^{(0)}(\mathbf{y}_{-i}).$$

Hence, we get

$$h_i^{(k+1)}(\mathbf{x}_{-i}) = f_i(h_i^{(k)}(\mathbf{x}_{-i}), \mathbf{x}_{-i}) \leq f_i(h_i^{(k)}(\mathbf{y}_{-i}), \mathbf{y}_{-i}) = h_i^{(k+1)}(\mathbf{y}_{-i}).$$

(d) As the sequence  $(h_i^{(k)}(\mathbf{x}_{-i}))_{k \in \mathbb{N}}$  is monotonically increasing and bounded from above by  $\mu \mathbf{f}_i$ , the sequence converges. Thus, for every  $\mathbf{x}$  the value

$$h_i(\mathbf{x}_{-i}) = \lim_{k \rightarrow \infty} h_i^{(k)}(\mathbf{x}_{-i})$$

is well-defined, i.e.,  $h_i$  is a map from  $[\mathbf{0}, \mu \mathbf{f}_{-i}]$  to  $[0, \mu \mathbf{f}_i]$ .

If  $f_i$  depends on at least one other variable except  $X_i$ , then  $h_i$  is a non-constant power series in this variable with non-negative coefficients. For  $\mathbf{x}_{-i} \in [\mathbf{0}, \mu \mathbf{f}_{-i})$  we thus always have

$$h_i(\mathbf{x}_{-i}) < h_i(\mu \mathbf{f}_{-i}) = \mu \mathbf{f}_i$$

as  $\mathbf{x}_{-i} \prec \mu \mathbf{f}_{-i}$ .

(e) This follows immediately from (b).

(f) As  $f_i$  is continuous, we have

$$f_i(h_i(\mathbf{x}_{-i}), \mathbf{x}_{-i}) = f_i(\lim_{k \rightarrow \infty} h_i^{(k)}(\mathbf{x}_{-i}), \mathbf{x}_{-i}) = \lim_{k \rightarrow \infty} h_i^{(k+1)}(\mathbf{x}_{-i}) = h_i(\mathbf{x}_{-i}),$$

where the last equality holds because of (b).

(g) Using induction similar to (a) replacing  $\mu \mathbf{f}$  by  $\mathbf{x}$ , one gets  $h_i^{(k)}(\mathbf{x}_{-i}) \leq x_i$  for all  $k \in \mathbb{N}$  as  $f_i(\mathbf{x}_{-i}) = x_i$ . Thus,  $h_i(\mathbf{x}_{-i}) \leq x_i$  follows similarly to (d).

(h) By definition, we have  $\mu \mathbf{f} = \lim_{k \rightarrow \infty} \mathbf{f}^k(\mathbf{0})$ . For  $k = 0$ , we have

$$(\mathbf{f}^0(\mathbf{0}))_i = 0 \leq f_i(0, \mu \mathbf{f}_{-i}) = h_i^{(0)}(\mu \mathbf{f}_{-i}).$$

We thus get by induction

$$(\mathbf{f}^{(k+1)}(\mathbf{0}))_i = f_i(\mathbf{f}^k(\mathbf{0})) \leq f_i(h_i^{(k)}(\mu \mathbf{f}_{-i}), \mu \mathbf{f}_{-i}) = h_i^{(k+1)}(\mu \mathbf{f}_{-i}).$$

Thus, we may conclude  $\mu \mathbf{f}_i \leq h_i(\mu \mathbf{f}_{-i})$ . As  $\mu \mathbf{f}_i = f_i(\mu \mathbf{f})$ , we get by virtue of (g) that  $h_i(\mu \mathbf{f}_{-i}) \leq \mu \mathbf{f}_i$ , too.

□

LEMMA 8.4.  $h_i$  is continuously differentiable with

$$\partial_{X_j} h_i(\mathbf{x}_{-i}) = \frac{\partial_{X_j} f_i(\mathbf{x})}{-\partial_{X_i} q_i(\mathbf{x})} = \frac{\partial_{X_j} q_i(\mathbf{x})}{-\partial_{X_i} q_i(\mathbf{x})} \text{ for } \mathbf{x} \in S_i \text{ and } j \neq i.$$

In particular,  $\partial_{X_j} h_i$  is monotonically increasing with  $\mathbf{x}$ .

*Proof.* By Lemma 8.2 the implicit function theorem is applicable for every  $\mathbf{x} \in S_i$ . We therefore find for every  $\mathbf{x} \in S_i$  a local parametrization  $h_{\mathbf{x}} : U \mapsto V$  with  $h_{\mathbf{x}}(\mathbf{x}_{-i}) = x_i$ . Thus  $h_{\mathbf{x}}(\mathbf{x}_{-i})$  is the least non-negative solution of  $q_i(X_i, \mathbf{x}_{-i}) = 0$ . By continuity of  $q_i$  it is now easily shown that for all  $\mathbf{y}_{-i} \in U$  it has to hold that  $h_{\mathbf{x}}(\mathbf{y}_{-i})$  is also the least non-negative solution of  $q_i(X_i, \mathbf{y}_{-i}) = 0$  (see below). By

uniqueness we therefore have  $h_{\mathbf{x}} = h_i$  and that  $h_i$  is continuously differentiable for all  $\mathbf{x}_{-i} \in [\mathbf{0}, \mu \mathbf{f}_{-i}]$ .

For every  $\mathbf{x}_{-i} \in [\mathbf{0}, \mu \mathbf{f}_{-i}]$  we can solve the (at most) quadratic equation  $q_i(X_i, \mathbf{x}_{-i}) = 0$ . We already know that  $h_i(\mathbf{x}_{-i})$  is the least non-negative solution of this equation. So, if there exists another solution, it has to be real, too.

Assume first that this equation has two distinct solutions for some fixed  $\mathbf{x}_{-i} \in [\mathbf{0}, \mu \mathbf{f}_{-i}]$ . Solving  $q_i(X_i, \mathbf{x}_{-i}) = 0$  thus leads to an expression of the form

$$\frac{-b(\mathbf{x}_{-i}) \pm \sqrt{b(\mathbf{x}_{-i})^2 - 4a \cdot c(\mathbf{x}_{-i})}}{2a}$$

for the solutions where  $b, c$  are (at most) quadratic polynomials in  $\mathbf{X}_{-i}$ ,  $c$  having non-negative coefficients, and  $a$  is a positive constant (leading coefficient of  $X_i^2$  in  $q_i(\mathbf{X})$ ). As  $b$  and  $c$  are continuous, the discriminant  $b(\cdot)^2 - 4a \cdot c(\cdot)$  stays positive for some open ball around  $\mathbf{x}_{-i}$  included inside of  $U$  (it is positive in  $\mathbf{x}_{-i}$  as we assume that we have two distinct solutions). By making  $U$  smaller, we may assume that  $U$  is this open ball. One of the two solutions has than to be the least non-negative solution. As  $h_{\mathbf{x}}$  is the least non-negative solution for  $\mathbf{x}_{-i}$ , and  $h_{\mathbf{x}}$  is continuous, this also has to hold for some open ball centered at  $\mathbf{x}_{-i}$ , wlog.  $U$  is this ball. So,  $h_{\mathbf{x}}$  and  $h_i$  coincide on  $U$ .

We turn to the case that  $q_i(X_i, \mathbf{x}_{-i}) = 0$  has only a single solution, i.e.  $h_i(\mathbf{x}_{-i})$ . Note that  $q_i(\mathbf{X})$  is linear in  $X_i$  if, and only if,  $q_i(X_i, \mathbf{x}_{-i})$  is linear in  $X_i$ . Obviously, if  $q_i$  linear in  $X_i$ , then  $h_i$  and  $h_{\mathbf{x}}$  coincide on  $U$ . Thus, consider the case that  $q_i(\mathbf{X})$  is quadratic in  $X_i$ , but  $q_i(X_i, \mathbf{x}_{-i})$  has only a single solution. This means that  $\mathbf{x}_{-i}$  is a root of the discriminant, i.e.  $b(\mathbf{x}_{-i}) - 4ac(\mathbf{x}_{-i}) = 0$ . As  $h_i(\mathbf{y}_{-i})$  is a solution of  $q_i(X_i, \mathbf{y}_{-i}) = 0$  for all  $\mathbf{y}_{-i} \in U$ , the discriminant is non-negative on  $U$ . If it equal to zero on  $U$ , then we again have that  $h_i$  is equal to  $h_{\mathbf{x}}$  on  $U$ . Therefore assume that is positive in some point of  $U$ . As the discriminant is continuous, the solutions change continuously with  $\mathbf{x}_{-i}$ . But this implies that for some  $\mathbf{y}_{-i} \in U$  there are at least two  $y_i, y_i^* \in V$  such that  $(\mathbf{y}_{-i}, y_i)$  and  $(\mathbf{y}_{-i}, y_i^*)$  are both located on the quadric  $q_i(\mathbf{X}) = 0$ . But this contradicts the uniqueness of  $h_{\mathbf{x}}$  guaranteed by the implicit function theorem.

Assume now that  $\mathbf{x} \in S_i$ . We then have

$$q_i(\mathbf{x}) = q_i(\mathbf{x}_{-i}, h_i(\mathbf{x}_{-i})) = 0,$$

or equivalently

$$f_i(\mathbf{x}_{-i}, h_i(\mathbf{x}_{-i})) = h_i(\mathbf{x}_{-i}).$$

Calculating the gradient of both in  $\mathbf{x}$  yields

$$f'_i(\mathbf{x}) \cdot \mathbf{p}'_i(\mathbf{x}_{-i}) = h'_i(\mathbf{x}_{-i}).$$

For the Jacobian of  $\mathbf{p}_i$  we obtain

$$\mathbf{p}'_i(\mathbf{x}_{-i}) = \begin{pmatrix} \mathbf{e}_1^\top \\ \vdots \\ \mathbf{e}_{i-1}^\top \\ h'_i(\mathbf{x}_{-i}) \\ \mathbf{e}_{i+1}^\top \\ \vdots \\ \mathbf{e}_n^\top \end{pmatrix}.$$

This leads to

$$\partial_{X_j} f_i(\mathbf{x}) + \partial_{X_i} f_i(\mathbf{x}) \cdot \partial_{X_j} h_i(\mathbf{x}_{-i}) = \partial_{X_j} h_i(\mathbf{x}_{-i})$$

which solved for  $\partial_{X_j} h_i$  yields

$$\partial_{X_j} h_i(\mathbf{x}_{-i}) = \frac{\partial_{X_j} f_i(\mathbf{x})}{-\partial_{X_i} q_i(\mathbf{x})}.$$

As  $\partial_{X_i} q_i(\mathbf{x}) < 0$  and both  $\partial_{X_j} f_i$  and  $\partial_{X_i} q_i$  monotonically increase with  $\mathbf{x}$ , it follows that  $\partial_{X_j} h_i$  also monotonically increases with  $\mathbf{x}$ . Finally, for  $j \neq i$  we have that  $\partial_{X_j} q_i = \partial_{X_j} f_i$  as  $q_i = f_i - X_i$ .  $\square$

LEMMA 8.8. *For all  $\mathbf{x} \in S_i$  we have*

$$\forall \mathbf{y} \in S_i \cap [\mathbf{x}, \mu \mathbf{f}] : q'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \leq 0.$$

*In particular*

$$\forall \mathbf{y} \in S_i \cap [\mathbf{x}, \mu \mathbf{f}] : y_i \geq x_i + \sum_{j \neq i} \partial_{X_j} h_i(\mathbf{x}_{-i}) \cdot (y_j - x_j).$$

*Proof.* Let  $\mathbf{x} \in S_i$ , i.e.  $f_i(\mathbf{x}) = x_i$ . We want to show that

$$q'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \leq 0$$

for all  $\mathbf{y} \in S_i \cap [\mathbf{x}, \mu \mathbf{f}]$ . As  $f_i$  is quadratic in  $\mathbf{X}$ , we may write

$$\begin{aligned} 0 &= q_i(\mathbf{y}) \\ &= -y_i + f_i(\mathbf{y}) \\ &= -y_i + \underbrace{f_i(\mathbf{x})}_{=x_i} + f'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + \underbrace{(\mathbf{y} - \mathbf{x})^\top \cdot A \cdot (\mathbf{y} - \mathbf{x})}_{\geq 0} \\ &\geq -y_i + x_i + f'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \\ &= f'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) - \mathbf{e}_i^\top \cdot (\mathbf{y} - \mathbf{x}) \\ &= q'_i(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \end{aligned}$$

where  $A$  is a symmetric square-matrix with non-negative components such that the quadratic terms of  $f_i$  are given by  $\mathbf{X}^\top A \mathbf{X}$ .

The second claim is easily obtained by solving this inequality for  $y_i$  and recalling that by Lemma 8.4 we have  $\partial_{X_j} h_i(\mathbf{x}_{-i}) = \frac{\partial_{X_j} q_i(\mathbf{p}_i(\mathbf{x}_{-i}))}{-\partial_{X_i} q_i(\mathbf{p}_i(\mathbf{x}_{-i}))}$  and  $\partial_{X_i} q_i(\mathbf{p}_i(\mathbf{x}_{-i})) < 0$ .  $\square$

PROPOSITION 8.9. *It holds*

$$\mathbf{x} \in R \Leftrightarrow \mathbf{x} \in [\mathbf{0}, \mu \mathbf{f}] \wedge \mathbf{q}(\mathbf{x}) \geq \mathbf{0}.$$

*Proof.* Let  $\mathbf{x} \in R$  and  $i \in \{1, \dots, n\}$ . Consider the function

$$g(t) := q_i(\mathbf{p}_i(\mathbf{x}_{-i}) + t\mathbf{e}_i).$$

As  $q_i$  is a quadratic polynomial in  $\mathbf{X}$  there exists a symmetric square-matrix  $A$  with non-negative entries, a vector  $\mathbf{b}$ , and a constant  $c$  such that

$$q_i(\mathbf{X}) = \mathbf{X}^\top A \mathbf{X} + \mathbf{b}^\top \mathbf{X} + c.$$



It then follows that

$$q_i(\mathbf{X} + \mathbf{Y}) = q_i(\mathbf{X}) + q'_i(\mathbf{X})\mathbf{Y} + \mathbf{Y}^\top \mathbf{A}\mathbf{Y}.$$

With  $q_i(\mathbf{p}_i(\mathbf{x}_{-i})) = 0$  this implies

$$g(t) = q'_i(\mathbf{p}_i(\mathbf{x}_{-i}))t\mathbf{e}_i + t^2 \underbrace{\mathbf{e}_i^\top \mathbf{A}\mathbf{e}_i}_{:=a \geq 0} = t \cdot (\partial_{X_i} q_i(\mathbf{p}_i(\mathbf{x}_{-i})) + a \cdot t).$$

As  $\mathbf{p}_i(\mathbf{x}_{-i}) \prec \mu \mathbf{f}$  ( $\mathbf{f}$  is strongly connected and  $\mathbf{x} \in [\mathbf{0}, \mu \mathbf{f}]$ ), we know that  $\partial_{X_i} q_i(\mathbf{p}_i(\mathbf{x}_{-i})) < 0$ . Thus,  $g(t)$  has at most two zeros, one at 0, the other for some  $t^* \geq 0$ .

For the direction ( $\Rightarrow$ ) we only have to show that  $x_i \leq h_i(\mathbf{x}_{-i})$  implies that  $q_i(\mathbf{x}) \geq 0$ . This now easily follows as  $x_i \leq h_i(\mathbf{x}_{-i})$  implies that there is a  $t' \leq 0$  with  $\mathbf{p}_i(\mathbf{x}_{-i}) + t'\mathbf{e}_i = \mathbf{x}$ . But for this  $t' \leq 0$  we have  $q_i(\mathbf{x}) = g(t') \geq 0$ .

Consider therefore the other direction ( $\Leftarrow$ ), that is  $\mathbf{x} \in [\mathbf{0}, \mu \mathbf{f}]$  with  $q_i(\mathbf{x}) \geq 0$ . Assume that  $\mathbf{x} \notin R$ , i.e., for at least one  $i$  we have  $x_i > h_i(\mathbf{x}_{-i})$ . As  $q_i(\mathbf{x}) \geq 0$  there has to be a  $t'' > 0$  with  $\mathbf{p}_i(\mathbf{x}_{-i}) + t''\mathbf{e}_i = \mathbf{x}$  and  $g(t'') \geq 0$ . This implies that  $a > 0$  has to hold as otherwise  $g(t)$  would be linear in  $t$  and negative for  $t > 0$ . But then the second root  $t^*$  of  $g(t)$  has to be positive. Set  $\mathbf{x}^* = \mathbf{p}_i(\mathbf{x}_{-i}) + t^*\mathbf{e}_i$  with  $q_i(\mathbf{x}^*) = 0$ , too.

A calculation similar to the one from above leads to

$$g(t + t^*) = q_i(\mathbf{x}^* + t\mathbf{e}_i) = t \cdot (\partial_{X_i} q_i(\mathbf{x}^*) + a \cdot t).$$

It follows that  $\partial_{X_i} q_i(\mathbf{x}^*)$  has to be greater than zero for  $-t^*$  to be a root (as  $a > 0$ ). But we have shown that  $\partial_{X_i} q_i(\mathbf{x}) < 0$  for all  $\mathbf{x} \in [\mathbf{0}, \mu \mathbf{f}]$ .  $\square$

LEMMA 8.11. *Let  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in [\mathbf{0}, \mu \mathbf{f}]$ . Then the matrix*

$$\begin{pmatrix} q'_1(\mathbf{x}^{(1)}) \\ \vdots \\ q'_n(\mathbf{x}^{(n)}) \end{pmatrix}$$

*is regular, i.e., the vectors  $\{q'_i(\mathbf{x}^{(i)}) \mid i = 1, \dots, n\}$  are linearly independent.*

*Proof.* Define  $\mathbf{x} \in [\mathbf{0}, \mu \mathbf{f}]$  by setting

$$x_i := \max\{x_i^{(j)} \mid j = 1, \dots, n\}.$$

We then have  $\mathbf{x}^{(i)} \leq \mathbf{x}$  for all  $i$ , and  $\mathbf{x} \prec \mu \mathbf{f}$ . As mentioned above, we therefore have that  $\mathbf{q}'(\mathbf{x})$  is regular with

$$\mathbf{q}'(\mathbf{x})^{-1} = - \sum_{k \in \mathbb{N}} \mathbf{f}'(\mathbf{x})^k.$$

As  $\mathbf{x}^{(i)} \leq \mathbf{x}$  it follows that

$$\begin{pmatrix} f'_1(\mathbf{x}^{(1)}) \\ \vdots \\ f'_n(\mathbf{x}^{(n)}) \end{pmatrix} \leq \mathbf{f}'(\mathbf{x}).$$

Hence, we also have

$$\sum_{k=0}^l \begin{pmatrix} f_1'(\mathbf{x}^{(1)}) \\ \vdots \\ f_n'(\mathbf{x}^{(n)}) \end{pmatrix}^k \leq \sum_{k=0}^l \mathbf{f}'(\mathbf{x})$$

implying that

$$\begin{pmatrix} f_1'(\mathbf{x}^{(1)}) \\ \vdots \\ f_n'(\mathbf{x}^{(n)}) \end{pmatrix}^* \quad \text{and, thus,} \quad \begin{pmatrix} q_1'(\mathbf{x}^{(1)}) \\ \vdots \\ q_n'(\mathbf{x}^{(n)}) \end{pmatrix}^{-1} \quad \text{exist.}$$

So, the vectors  $\{q_1'(\mathbf{x}^{(1)}), \dots, q_n'(\mathbf{x}^{(n)})\}$  have to be linearly independent.  $\square$

THEOREM 8.13. *We have*

$$\mathbf{x} \leq \mathcal{T}(\mathbf{x}) \quad \text{and} \quad \mathcal{N}(\mathbf{x}) \leq \mathcal{T}(\mathbf{x}) \leq \mu \mathbf{f}.$$

Further, for  $\mathbf{y} \in R$  with  $\mathbf{x} \leq \mathbf{y}$

$$\mathcal{T}(\mathbf{x}) \leq \mathcal{T}(\mathbf{y}).$$

*Proof.* As  $\mathbf{x} \leq \mathcal{N}(\mathbf{x})$  holds, we only need to show that  $\mathcal{N}(\mathbf{x}) \leq \mathcal{T}(\mathbf{x}) \leq \mu \mathbf{f}$ . We refer the reader to the proof of definition B.2 and Theorem B.3 for this.

We turn to the monotonicity of  $\mathcal{T}$ . Let  $\mathbf{y} \in R$  with  $\mathbf{x} \leq \mathbf{y}$ . Assume first that  $\mathbf{x}$  and  $\mathbf{y}$  are located on the surface  $S_i$ , i.e.

$$h_i(\mathbf{x}_{-i}) = x_i \quad \text{and} \quad h_i(\mathbf{y}_{-i}) = y_i.$$

The tangent  $T_i|_{\mathbf{x}}$  at  $S_i$  in  $\mathbf{x}$  is spanned by the partial derivatives of  $\mathbf{p}_i$  in  $\mathbf{x}$ . The part  $T_i|_{\mathbf{x}} \cap [\mathbf{x}, \mu \mathbf{f}]$  relevant for  $\mathcal{T}(\mathbf{x})$  can therefore be parameterized by

$$\mathbf{x} + \sum_{j \neq i} \partial_{X_j} \mathbf{p}_i(\mathbf{x}) \cdot (u_j - x_j) \quad \text{with} \quad \mathbf{u}_{-i} \in [\mathbf{x}_{-i}, \mu \mathbf{f}_{-i}].$$

Similarly for  $T_i|_{\mathbf{y}}$ .

In particular, for  $\mathbf{u}_{-i} \in [\mathbf{y}_{-i}, \mu \mathbf{f}_{-i}]$  both points on the tangents defined by  $\mathbf{u}_{-i}$  differ only in the  $i$ th coordinate being (the remaining coordinates are simply  $\mathbf{u}_{-i}$ )

$$t_{\mathbf{y}} = y_i + \sum_{j \neq i} \partial_{X_j} h_i(\mathbf{y}) \cdot (u_j - y_j), \quad \text{resp.} \quad t_{\mathbf{x}} = x_i + \sum_{j \neq i} \partial_{X_j} h_i(\mathbf{x}) \cdot (u_j - x_j).$$

By Lemma 8.8 we have

$$y_i \geq x_i + \sum_{j \neq i} \partial_{X_j} h_i(\mathbf{x}) \cdot (y_j - x_j).$$

From Lemma 8.4 it follows that  $\partial_{X_j} h_i(\mathbf{y}) \geq \partial_{X_j} h_i(\mathbf{x})$ . Thus  $t_{\mathbf{y}} \geq t_{\mathbf{x}}$  immediately follows.

Now for  $\mathbf{x}, \mathbf{y} \in R$  with  $\mathbf{x} \leq \mathbf{y}$  we can apply this result to the tangents at  $S_i$  in  $\mathbf{p}_i(\mathbf{x}_{-i})$ , resp.  $\mathbf{p}_i(\mathbf{y}_{-i})$ , and  $\mathcal{T}(\mathbf{x}) \leq \mathcal{T}(\mathbf{y})$  follows.  $\square$

DEFINITION B.2. Let  $\mathbf{x} \in R$ . For  $i = 1, \dots, n$  fix some  $\eta_i \in [x_i, h_i(\mathbf{x}_{-i})]$ , and set  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ . We then let  $\mathcal{T}_\boldsymbol{\eta}(\mathbf{x})$  denote the solution of

$$q'_i((\mathbf{x}_{-i}, \eta_i))(\mathbf{X} - (\mathbf{x}_{-i}, \eta_i)) = -q_i((\mathbf{x}_{-i}, \eta_i)) \quad (i = 1, \dots, n).$$

Note that we have  $\mathcal{N}(\mathbf{X}) = \mathcal{T}_\mathbf{x}(\mathbf{X})$ , and  $\mathcal{T}(\mathbf{X}) = \mathcal{T}_{(h_1(\mathbf{x}_{-1}), \dots, h_n(\mathbf{x}_{-n}))}(\mathbf{X})$ .

THEOREM B.3. Let  $\mathbf{x} \in R$ . For  $i = 1, \dots, n$  fix some  $\eta_i \in [x_i, h_i(\mathbf{x}_{-i})]$ , and set  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ . We then have

$$\mathbf{x} \leq \mathcal{N}(\mathbf{x}) \leq \mathcal{T}_\boldsymbol{\eta}(\mathbf{x}) \leq \mathcal{T}(\mathbf{x}) \leq \mu \mathbf{f}$$

*Proof.* Set

$$\boldsymbol{\pi}_i := (\mathbf{x}_{-i}, \eta_i) \text{ and } \mathbf{h} := (h_1(\mathbf{x}_{-1}), \dots, h_n(\mathbf{x}_{-n})).$$

We first show that  $\mathbf{x} \leq \mathcal{T}_\boldsymbol{\eta}(\mathbf{x})$ :

$$\begin{aligned} \mathcal{T}_\boldsymbol{\eta}(\mathbf{x}) &= (q'_i(\boldsymbol{\pi}_i))_{i=1, \dots, n}^{-1} \cdot (q'_i(\boldsymbol{\pi}_i) \cdot \boldsymbol{\pi}_i - q_i(\boldsymbol{\pi}_i))_{i=1, \dots, n} \\ &= (f'_i(\boldsymbol{\pi}_i))_{i=1, \dots, n}^* \cdot (-q'_i(\boldsymbol{\pi}_i) \cdot \boldsymbol{\pi}_i + q_i(\boldsymbol{\pi}_i))_{i=1, \dots, n} \\ &= (f'_i(\boldsymbol{\pi}_i))_{i=1, \dots, n}^* \cdot (-q'_i(\boldsymbol{\pi}_i) \cdot (\mathbf{x} + (\eta_i - x_i) \cdot \mathbf{e}_i) + q_i(\boldsymbol{\pi}_i))_{i=1, \dots, n} \\ &= \underbrace{(f'_i(\boldsymbol{\pi}_i))_{i=1, \dots, n}^*}_{\geq 0 \text{ in every comp.}} \cdot \underbrace{(-q'_i(\boldsymbol{\pi}_i) \cdot \mathbf{x} - \underbrace{\partial_{X_i} q_i(\boldsymbol{\pi}_i)}_{< 0} \cdot \underbrace{(\eta_i - x_i)}_{\geq 0} + \underbrace{q_i(\boldsymbol{\pi}_i)}_{\geq 0})}_{i=1, \dots, n} \\ &\geq \mathbf{x}. \end{aligned}$$

$\mathcal{T}_\boldsymbol{\eta}(\mathbf{x})$  is by definition the (unique) solution of the equation system defined by

$$q'_i(\boldsymbol{\pi}_i)(\mathbf{X} - \boldsymbol{\pi}_i) = -q_i(\boldsymbol{\pi}_i) \quad (i = 1, \dots, n).$$

As  $\mathcal{T}_\boldsymbol{\eta}(\mathbf{x}) \geq \mathbf{x}$  we can also consider this system with the origin of the coordinate system moved into  $\mathbf{x}$ , i.e.

$$q'_i(\boldsymbol{\pi}_i)(\mathbf{X} + \mathbf{x} - \boldsymbol{\pi}_i) = -q_i(\boldsymbol{\pi}_i) \quad (i = 1, \dots, n).$$

We show that this system is equivalent to an SPP. For this, we solve these equations for  $X_i$ :

$$\begin{aligned} q'_i(\boldsymbol{\pi}_i)(\mathbf{X} + \mathbf{x} - \boldsymbol{\pi}_i) &= -q_i(\boldsymbol{\pi}_i) \\ \Leftrightarrow q'_i(\boldsymbol{\pi}_i)\mathbf{X} &= -q_i(\boldsymbol{\pi}_i) + q'_i(\boldsymbol{\pi}_i) \underbrace{(\boldsymbol{\pi}_i - \mathbf{x})}_{= (\eta_i - x_i) \cdot \mathbf{e}_i} \\ \Leftrightarrow X_i &= \sum_{j \neq i} \frac{\partial_{X_j} q_i(\boldsymbol{\pi}_i)}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)} \cdot X_j + \frac{q_i(\boldsymbol{\pi}_i)}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)} + (\eta_i - x_i). \end{aligned}$$

Again, we have  $\partial_{X_i} q_i(\boldsymbol{\pi}_i) < 0 \leq \partial_{X_j} q_i(\boldsymbol{\pi}_i)$  as  $\boldsymbol{\pi}_i \in R$ , and  $q'_i(\boldsymbol{\pi}_i)$  monotonically increases with  $\eta_i$ . Hence, the above linear equation for  $X_i$  is indeed a polynomial with non-negative coefficients. Denote by  $\mathbf{f}_\boldsymbol{\eta}$  the SPP defined by these linear equations. We then have  $\mu \mathbf{f}_\boldsymbol{\eta} = \mathcal{T}_\boldsymbol{\eta}(\mathbf{x}) - \mathbf{x}$  as the above equation system has  $\mathcal{T}_\boldsymbol{\eta}(\mathbf{x}) - \mathbf{x} \geq \mathbf{0}$  as its unique solution. Further, we know that the Kleene sequence  $(\mathbf{f}_\boldsymbol{\eta}^k(\mathbf{0}))_{k \in \mathbb{N}}$  converges to  $\mu \mathbf{f}_\boldsymbol{\eta}$ . We show that all coefficients of  $\mathbf{f}_\boldsymbol{\eta}$  increase with  $\boldsymbol{\eta} \rightarrow \mathbf{h}$ . This is straight-forward for

$$\frac{\partial_{X_j} q_i(\boldsymbol{\pi}_i)}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)}$$

as  $\partial_{X_i} q_i(\boldsymbol{\pi}_i) < 0 \leq \partial_{X_j} q_i(\boldsymbol{\pi}_i)$ , and all these terms increase with  $\eta_i \rightarrow h_i(\mathbf{x}_{-i})$ . Consider therefore

$$0 \geq \frac{q_i(\boldsymbol{\pi}_i)}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)} + (\eta_i - x_i) = \frac{q_i(\boldsymbol{\pi}_i) - \partial_{X_i} q_i(\boldsymbol{\pi}_i)(\eta_i - x_i)}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)}.$$

We show that this term increases with  $\eta_i$ . Set  $\delta_i := \eta_i - x_i$ . We can find a non-negative, symmetric square-matrix  $A$ , a vector  $\mathbf{b}$ , and constant  $c$  such that

$$q_i(\mathbf{X}) = \mathbf{X}^\top A \mathbf{X} + \mathbf{b}^\top \mathbf{X} + c \text{ and } q_i'(\mathbf{X}) = 2\mathbf{X}^\top A + \mathbf{b}^\top.$$

As  $\boldsymbol{\pi}_i = \mathbf{x} + \delta_i \mathbf{e}_i$  we have

$$q_i(\boldsymbol{\pi}_i) = q_i(\mathbf{x} + \delta_i \mathbf{e}_i) = q_i(\mathbf{x}) + \partial_{X_i} q_i(\mathbf{x}) \delta_i + \delta_i^2 A_{ii},$$

and

$$\partial_{X_i} q_i(\boldsymbol{\pi}_i) \cdot \delta_i = q_i'(\mathbf{x} + \delta_i \mathbf{e}_i) \delta_i \mathbf{e}_i = \partial_{X_i} q_i(\mathbf{x}) \delta_i + 2\delta_i^2 A_{ii}.$$

This leads to

$$\frac{q_i(\boldsymbol{\pi}_i) - \partial_{X_i} q_i(\boldsymbol{\pi}_i) \delta_i}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)} = \frac{q_i(\mathbf{x}) - \delta_i^2 A_{ii}}{-\partial_{X_i} q_i(\mathbf{x}) - 2\delta_i A_{ii}}.$$

Deriving this w.r.t.  $\delta_i$  yields:

$$\begin{aligned} & \frac{-2A_{ii}\delta_i}{-\partial_{X_i} q_i(\mathbf{x}) + 2A_{ii}\delta_i} - \frac{q_i(\mathbf{x}) - A_{ii}\delta_i^2}{(-\partial_{X_i} q_i(\mathbf{x}) - 2A_{ii}\delta_i)^2} (-2A_{ii}) \\ &= \frac{2A_{ii}\partial_{X_i} q_i(\mathbf{x})\delta_i + 4A_{ii}^2\delta_i^2 + 2A_{ii}q_i(\mathbf{x}) - 2A_{ii}^2\delta_i^2}{(-\partial_{X_i} q_i(\mathbf{x}) - 2A_{ii}\delta_i)^2} \\ &= 2A_{ii} \frac{A_{ii}\delta_i^2 + \partial_{X_i} q_i(\mathbf{x})\delta_i + q_i(\mathbf{x})}{(-\partial_{X_i} q_i(\mathbf{x}) - 2A_{ii}\delta_i)^2} \\ &= 2A_{ii} \frac{q_i(\boldsymbol{\pi}_i)}{(-\partial_{X_i} q_i(\boldsymbol{\pi}_i))^2}. \end{aligned}$$

As  $q_i(\boldsymbol{\pi}_i) \geq 0$  and  $A_{ii} \geq 0$ , it follows that

$$\frac{q_i(\boldsymbol{\pi}_i)}{-\partial_{X_i} q_i(\boldsymbol{\pi}_i)} + (\eta_i - x_i)$$

increases with  $\eta_i \rightarrow h_i(\mathbf{x}_{-i})$ . Thus, all coefficients of  $\mathbf{f}_\eta$  increase with  $\eta_i \rightarrow h_i(\mathbf{x}_{-i})$ , and so for any  $\boldsymbol{\eta}' \in [\boldsymbol{\eta}, \mathbf{h}]$  it follows that

$$\mathbf{f}_\eta(\mathbf{y}) \leq \mathbf{f}_{\boldsymbol{\eta}'}(\mathbf{y}) \text{ for all } \mathbf{y} \geq \mathbf{0},$$

and

$$\mathcal{T}_\eta(\mathbf{x}) - \mathbf{x} = \mu \mathbf{f}_\eta \leq \mu \mathbf{f}_{\boldsymbol{\eta}'} = \mathcal{T}_{\boldsymbol{\eta}'}(\mathbf{x}) - \mathbf{x}.$$

As  $\mathcal{N}(\mathbf{X}) = \mathcal{T}_\mathbf{x}(\mathbf{X})$  and  $\mathcal{T}(\mathbf{X}) = \mathcal{T}_\mathbf{h}(\mathbf{X})$  we may therefore conclude that

$$\mathcal{N}(\mathbf{x}) \leq \mathcal{T}_\eta(\mathbf{x}) \leq \mathcal{T}_{\boldsymbol{\eta}'}(\mathbf{x}) \leq \mathcal{T}(\mathbf{x}).$$

It remains to show that  $\mathcal{T}(\mathbf{x}) \leq \mu \mathbf{f}$ . This is equivalent to showing that  $\mu \mathbf{f}_\mathbf{h} \leq \mu \mathbf{f} - \mathbf{x}$ . For  $\mathbf{f}_\mathbf{h}(\mathbf{X})$  we have by definition and Lemma 8.4

$$(\mathbf{f}_\mathbf{h}(\mathbf{X}))_i = \sum_{j \neq i} \frac{\partial_{X_j} q_i(\mathbf{p}_i(\mathbf{x}_{-i}))}{-\partial_{X_i} q_i(\mathbf{p}_i(\mathbf{x}_{-i}))} X_j + (h_i(\mathbf{x}_{-i}) - x_i) = \sum_{j \neq i} \partial_{X_j} h_i(\mathbf{x}_{-i}) X_j + (h_i(\mathbf{x}_{-i}) - x_i).$$

By virtue of Lemma 8.8 it follows that  $\mu\mathbf{f}$  is above all the tangents, i.e.

$$\mathbf{f}_h(\mu\mathbf{f} - \mathbf{x}) \leq \mu\mathbf{f} - \mathbf{x}.$$

By monotonicity of  $\mathbf{f}_h$  we also have

$$\mathbf{f}_h(\mathbf{0}) \leq \mathbf{f}_h(\mu\mathbf{f} - \mathbf{x}).$$

A straight-forward induction therefore shows that

$$\mathbf{f}_h^k(\mathbf{0}) \leq \mu\mathbf{f} - \mathbf{x} \quad (\forall k \in \mathbb{N}),$$

and, thus,

$$\mathcal{T}(\mathbf{x}) - \mathbf{x} = \mu\mathbf{f}_h \leq \mu\mathbf{f} - \mathbf{x}.$$

□

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