

# Synthesis Rules for Petri Nets, and How they Lead to New Results \*

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## Abstract

Three kits of rules for top-down synthesis of Petri nets are introduced. The properties and expressive power of the kits are compared. They are then used to characterise the class of structurally live Free Choice nets by means of the rank of the incidence matrix.

## Introduction

It is a well learnt lesson that systems, even not very large ones, can be successfully built only by means of disciplined design. A particular implementation of this discipline is the top-down paradigm: systems are designed through stepwise refinement of a simple initial system. Correctness is proved using induction: the initial system is shown to meet the specification, and the refinements are shown to preserve it (i.e if a system in the refinement sequence meets the specification, so does its successor). This process is well understood for sequential systems, but not so much for concurrent ones. Petri nets provide a formal framework where this problem can be addressed.

In a previous paper [3], some research along this line was carried out. Two requirements which are part of the specification of many systems were selected: absence of global or partial deadlocks and absence of overflows in finite stores. In the Petri net formalism they correspond to the properties of liveness and boundedness. On the other hand, also the class of systems was restricted to the ones modelled by means of Free Choice nets, a class of nets which permits to represent both concurrency and nondeterminism, but constraints the interplay between both. The goal was to provide a sound and complete kit of refinements rules, i.e a kit of rules which preserves liveness and boundedness, and allows one to synthesise every live and bounded Free Choice net. It turned out that two refinement rules sufficed.

The technical details in [3] possibly hid the simplicity of the final result. The first goal of this paper is to overcome this problem: the kit of rules is presented here again, hopefully

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in a readable form, and is situated in the context of other new kits, whose properties and relationships are considered. The second goal is to show how synthesis procedures can lead to a deeper understanding of the class of nets they synthesise: a simple algebraic characterisation of structural liveness is given for the class of Free Choice nets that can be decomposed into State Machines (Free Choice nets in which concurrency is due to synchronous communication between sequential processes by rendez-vous). More precisely, it is shown that structural liveness (the existence of a marking for which the net is live) can be decided calculating the rank of the incidence matrix of the net. From this result, Hack's duality theorem and a polynomial algorithm for deciding liveness can be easily derived.

The paper is organised as follows: section 1 contains basic definitions. In section 2, three sets of refinement rules for top-down synthesis of nets are described, the last one being the one introduced in [3] and mentioned above, and their properties discussed (this part can be considered, up to a certain extent, a survey). The algebraic characterisation is proved in section 3. Finally, section 4 shows the two consequences mentioned above.

## 1 Basic definitions

$\mathbf{N}$  denotes the set  $\mathbf{N} = \{0, 1, 2, \dots\}$ .  $\mathbf{Z}$  is the set of integers  $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ .  $\mathbf{Q}$  is the set of rational numbers. The cardinality of a set  $X$  is denoted  $|X|$ . A *class* is a set whose elements are also sets.

### 1.1 Nets

A *net* is a triple  $N = (S, T, F)$  where

- $S \cap T = \emptyset$
- $F \subseteq (S \times T) \cup (T \times S)$ .

The elements of  $S$ ,  $T$  and  $F$  are called *places*, *transitions* and *arcs*, respectively. Places and transitions are called generically *nodes*. We assume that  $S$  and  $T$  are totally ordered, and denote  $n = |S|$ ,  $m = |T|$ .  $n$  and  $m$  inherit the subscripts of the net they refer to. The *Pre-set*  ${}^{\circ}x$  of  $x \in S \cup T$  is given by  ${}^{\circ}x = \{y \in S \cup T \mid (y, x) \in F\}$ . The *Post-set*  $x^{\circ}$  of  $x \in S \cup T$  is given by  $x^{\circ} = \{y \in S \cup T \mid (x, y) \in F\}$ . We also define for  $X \subseteq S \cup T$

$${}^{\circ}X = \bigcup_{x \in X} {}^{\circ}x \quad X^{\circ} = \bigcup_{x \in X} x^{\circ}$$

If a node  $x$  belongs to more than one net, and there is ambiguity about which net the dot  ${}^{\circ}$  refers to, the name of the net is added as a subscript, unless it is  $N$ :  $({}^{\circ}x)_N$  denotes the Pre-set of  $x$  in  $N$ .

The operation  $\cup$  on sets is extended to nets in the natural way. Given two nets  $N_1 = (S_1, T_1, F_1)$ ,  $N_2 = (S_2, T_2, F_2)$  we define

$$N_1 \cup N_2 = (S_1 \cup S_2, T_1 \cup T_2, F_1 \cup F_2)$$

The intersection of nets is defined analogously. Notice that both the union and the intersection of two nets is a net.

$W$  denotes the characteristic function of  $F$ . The matrix  $C = \|c_{ij}\|$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  with

$$c_{ij} = W(t_j, s_i) - W(s_i, t_j)$$

is called the *incidence matrix* of  $N$ . That is, to each place of the net corresponds a row of the incidence matrix. Risking confusion, we denote this row with the name of the place. The rank of a matrix  $A$  (i.e the maximal number of linearly independent rows) is denoted by  $r(A)$ .

A *path* of a net  $N = (S, T, F)$  is an alternating sequence  $(x_1, f_1, x_2, \dots, f_{r-1}, x_r)$  of elements of  $S \cup T$  and  $F$  such that  $\forall i, 1 \leq i \leq r-1: f_i = (x_i, x_{i+1})$ . A path with arcs in a certain set  $F' \subseteq F$  is called an  $F'$ -path.

Given a net  $N = (S, T, F)$ , the net  $N^{-d} = (T, S, F^{-1})$  is called the *reverse-dual* net of  $N$ . It is easy to see that if  $C$  is the incidence matrix of  $N$ , then  $-C^T$  is the incidence matrix of  $N^{-d}$ .

A net  $N = (S, T, F)$  is called an *S-graph* iff  $\forall t \in T: |{}^*t| = 1 = |t^*|$ .  $N$  is called a *T-graph* iff  $\forall s \in S: |{}^*s| = 1 = |s^*|$ .  $N$  is called *free choice* iff  $\forall s \in S: |s^*| > 1 \Rightarrow {}^*(s^*) = \{s\}$ .

A net  $N' = (S', T', F')$  is a *subnet* of  $N$ , denoted  $N' \subseteq N$ , iff

$$S' \subseteq S \quad T' \subseteq T \quad F' = F \cap ((S' \times T') \cup (T' \times S'))$$

A place  $s' \in S'$  is a *way-in* (*way-out*) place of  $N' \subseteq N$  iff  ${}^*s' \not\subseteq T'$  ( $s'^* \not\subseteq T'$ ).  $N' \subseteq N$  is an *S-component* (*T-component*) of  $N$  iff  $N'$  is a strongly connected *S-graph* (*T-graph*) and  $T' = {}^*S' \cap S'^*$  ( $S' = {}^*T' \cup T'^*$ ).

$N$  is said to be *State Machine Decomposable* (*Marked Graph Decomposable*) iff there is a set  $\{N_1, \dots, N_r\}$  of *S-components* (*T-components*) of  $N$  such that

$$N = \bigcup_{i=1}^r N_i$$

State Machine Decomposable will be shortened to SMD.

## 1.2 Place/transition nets or Petri nets

A function  $M: S \rightarrow \mathbb{N}$  is called a *marking*. Markings are also represented in vector form: the  $i$ th component of the vector corresponds to  $M(s_i)$ .

A *Place/Transition net* or *Petri net*, is a pair  $(N, M_0)$  where  $N$  is a net and  $M_0$  is a marking called *initial marking*. A transition  $t \in T$  is *enabled* at a marking  $M$  iff  $\forall s \in {}^*t: M(s) > 0$ . If  $t \in T$  is enabled at a marking  $M$  then  $t$  may *occur* yielding a new marking  $M'$  given by

$$\forall s \in S: M'(s) = M(s) - W(s, t) + W(t, s)$$

$M[t]M'$  denotes the fact that  $M'$  is reached from  $M$  by the occurrence of  $t$ .

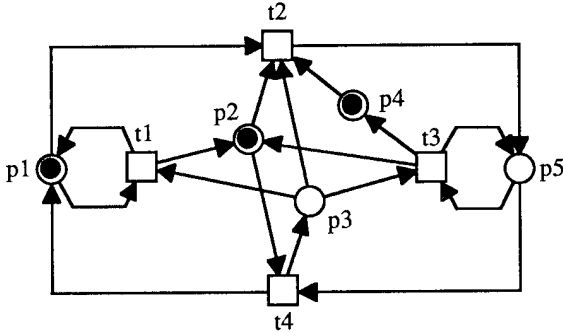


Figure 1: A live and bounded Petri net whose underlying net is not structurally bounded

A sequence of transitions,  $\sigma = t_1 t_2 \dots t_r$ , is a *transition sequence* of  $(N, M_0)$  iff there exists a sequence  $M_0 t_1 M_1 t_2 M_2 \dots t_r M_r$  such that  $\forall i, 1 \leq i \leq r: M_{i-1} [t_i] M_i$ . The marking  $M_n$  is said to be *reachable* from  $M_0$  by the occurrence of  $\sigma$ . The set of reachable markings of  $(N, M_0)$  is denoted by  $R(N, M_0)$ .

A Petri net  $(N, M_0)$  is *k-bounded* iff  $\forall s \in S, \forall M \in R(N, M_0): M(s) \leq k$ .  $(N, M_0)$  is *bounded* iff  $\exists k \in \mathbf{N}: (N, M_0)$  is *k-bounded*. A net  $N$  is *structurally bounded* iff  $\forall M_0 \exists k \in \mathbf{N}: (N, M_0)$  is bounded.

A transition  $t \in T$  is *live* in  $(N, M_0)$  iff  $\forall M \in R(N, M_0) \exists M' \in R(N, M): M'$  enables  $t$ .  $(N, M_0)$  is *live* iff all  $t \in T$  are live.  $N$  is *structurally live* iff  $\exists M_0: (N, M_0)$  is live. Structurally live and structurally bounded is shortened to SL&SB.

It follows from the definition that if  $N$  is SL&SB, then there is a marking  $M_0$  that makes  $(N, M_0)$  live and bounded. But the converse is not true. The Petri net of figure 1, taken from [8], is an example. It is live and bounded for the given marking, but unbounded for the marking (11110) (firing  $t_2 t_3 t_3 t_4$  the marking (11120)  $>$  (11110) is reached). Hence, the underlying net is not structurally bounded.

## 2 Refinement kits

Top-down synthesis of nets is performed starting from a very simple net to which refinement rules are applied stepwise. Given a class of rules (a *kit*), the nets that can be produced applying a finite number of times the elements of the kit is the class of nets generated by these rules. In this section we first formalise these concepts. Then we introduce three different refinement kits, together with their properties. The seed of the synthesis procedure is called *initial net*.

### Definition 2.1 Initial net

The net  $N_0 = (\{s\}, \{t\}, \{(s, t), (t, s)\})$  is called *initial net*. ■ 2.1

A refinement rule allows us to transform a net into another one, which under some criterion is considered more complex than the old one. This is adequately represented by means of an antisymmetric binary relation.

### Definition 2.2 Refinement rules

A refinement rule  $R$  is a binary antisymmetric relation on the class of nets  $\mathcal{N}$ . Given  $(N, \hat{N}) \in R$ ,  $N$  is called *source net* and  $\hat{N}$  *target net*. A class  $\{R_1, \dots, R_a\}$  of refinement rules is called a *refinement kit*. The class of nets *produced* by  $\{R_1, \dots, R_a\}$ , denoted  $\mathcal{N}(R_1, \dots, R_a)$ , is the smallest class of nets given by:

1.  $N_0 \in \mathcal{N}(R_1, \dots, R_a)$  (the initial net is produced by the kit)
2. If  $N \in \mathcal{N}(R_1, \dots, R_a)$  and  $\exists i, 1 \leq i \leq a : (N, \hat{N}) \in R_i$ , then  $\hat{N} \in \mathcal{N}(R_1, \dots, R_a)$  (i.e. if  $N$  is produced by the kit and  $\hat{N}$  is obtained by applying one of the rules of the kit to  $N$ , then  $\hat{N}$  is produced by the kit).

A sequence of nets  $(N_i)$ ,  $0 \leq i \leq r$ , where  $r \in \mathbb{N}$ , such that  $N_0$  is the initial net and

$$\forall i, 0 \leq i \leq (r-1) : (N_i, N_{i+1}) \in R_j, 1 \leq j \leq a$$

is called a *synthesis sequence* of  $N_r$  in  $\mathcal{N}(R_1, \dots, R_a)$ . ■ 2.2

We can now introduce the three kits we deal with in this paper. Each of them is composed by two rules.

**The SMD kit** The name of this kit is due to the fact that it produces all SMD nets (as well as others, but we are not interested in this). The other kits will be obtained taking subrelations of the rules of this kit, and therefore will generate smaller classes of nets.

The two rules  $R_1, R_2$  can be explained very easily in a graphical way.  $R_1$  consists of the addition of a new place to a net, with the condition that the new place must have at least one input arc and one output arc. Figure 2 shows an example, where the new place and its input and output arcs are printed in boldface.

$R_2$  consists of the substitution of a place by a connected  $S$ -graph. Two possible substitutions of the place  $\hat{s}$  of figure 2 are shown in figure 3 (the  $S$ -graphs are printed in boldface). Nevertheless, we impose two further conditions: one on the substituted place  $s$  and other on the  $S$ -graph  $N' = (S', T', F')$  substituting it.

First,  $s$  must satisfy  $\forall t \in {}^*s : |t^*| > 1$  and  $\forall t \in s^* : |t^*| > 1$ , except maybe at the very beginning, when refining the only place of the initial net.

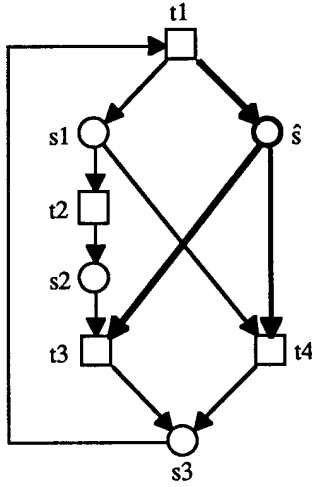
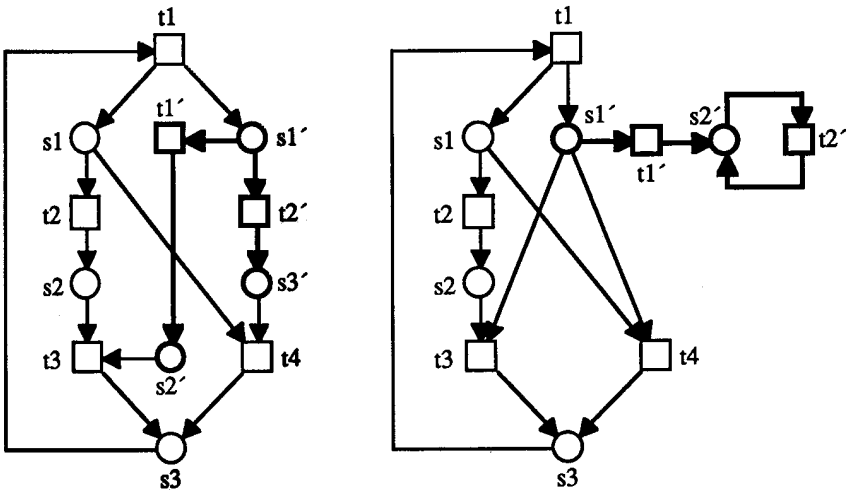
Second, every place of  $N'$  must be contained in an  $F'$ -path starting at a way-in place and ending at a way-out place of  $N'$ .

The substitution on the left of figure 3 is a legal one, since it satisfies both conditions. The substitution on the right does not satisfy the second.  $s'_1$  is the only way-in and the only way-out place of the  $S$ -graph. Nevertheless, no path of the  $S$ -graph starting and ending at  $s'_1$  contains  $s'_2$ .

This second condition is included for the following reason: it is easy to see that if the source net is strongly connected, the substitutions leading to a strongly connected target net are exactly the ones that satisfy it.

The formal definitions of the two rules are given next.

Let  $N = (S, T, F)$ ,  $\hat{N} = (\hat{S}, \hat{T}, \hat{F})$  be two nets.

Figure 2: The refinement rule  $R_1$ Figure 3: Two possible substitutions of the place  $\hat{s}$  of figure 2 by an  $S$ -graph. Only the left one is allowed by rule  $R_2$

**Rule 1**  $(N, \hat{N}) \in R_1$  iff:

1.  $\hat{S} = S \cup \hat{s}$ , where  $\hat{s} \notin S$ , and  $\hat{T} = T$
2.  $\hat{F} = F \cup F_{\hat{s}}$ , where  $F_{\hat{s}} \subseteq (\{\hat{s}\} \times T) \cup (T \times \{\hat{s}\})$  and  $(\bullet \hat{s})_{\hat{N}} \neq \emptyset \neq (\hat{s} \bullet)_{\hat{N}}$  ■ R1

**Definition 2.3** Let  $N = (S, T, F)$ ,  $N' = (S', T', F')$  be two nets, and  $s \in S$ . It is said that the net  $\hat{N} = (\hat{S}, \hat{T}, \hat{F})$  is obtained *replacing*  $s$  by  $N'$  in  $N$  iff

1.  $\hat{S} = (S - \{s\}) \cup S'$
2.  $\hat{T} = T \cup T'$
3.  $\hat{F} = \bar{F} \cup F' \cup F_s$ , where
  - $\bar{F} = F \cap ((\hat{S} \times \hat{T}) \cup (\hat{T} \times \hat{S}))$  (arcs remaining in  $N$  after removing  $s$ )
  - $F_s$  is obtained in the following way:  
for each arc  $f \in F \cap ((\{s\} \times T) \cup (T \times \{s\}))$ : select arbitrarily a place  $s' \in S'$ , replace  $s$  by  $s'$  and add the resulting arc to  $F_s$   
(arcs replacing the input and output arcs of  $s$ )

■ 2.3

**Rule 2**  $(N, \hat{N}) \in R_2$  iff there exist  $s \in S$  and an  $S$ -graph  $N' = (S', T', F') \subseteq \hat{N}$  satisfying:

1.  $N = N_0 \vee (\forall t \in \bullet s: |t^\bullet| > 1 \wedge \forall t \in s^\bullet: |t| > 1)$
2.  $\hat{N}$  is obtained replacing  $s$  by  $N'$  in  $N$
3.  $\forall s' \in S'$ : there exists an  $F'$ -path  $(s'_{in}, \dots, s', \dots, s'_{out})$ , where  $s'_{in}$  and  $s'_{out}$  are a way-in and a way-out place of  $N'$ , respectively. ■ R2

**Remark 2.4**

1. It is easy to see that, when applying  $R_2$ ,  $s = \sum_{s' \in S'} s'$  (where, as was announced in the past section, we identify a place with its corresponding row in the incidence matrix).
2. The nets of  $\mathcal{N}(R_1, R_2)$  are strongly connected. This can be proved inductively: the initial net is strongly connected and the two rules preserve strong connectedness.
3. Notice that  $R_2$  always operates on the places added by means of  $R_1$ , except when it is applied at  $N_0$ . The reason is that places not added by  $R_1$  (and not the initial place) must come from the refinement of a place by means of  $R_2$ . Then they belong to a subnet that is an  $S$ -graph, and therefore have one single input transition or one single output transition in the net. But in this case they do not satisfy condition 1 for the application of  $R_2$ .

■ 2.4

Now we state the property that gives name to the kit.

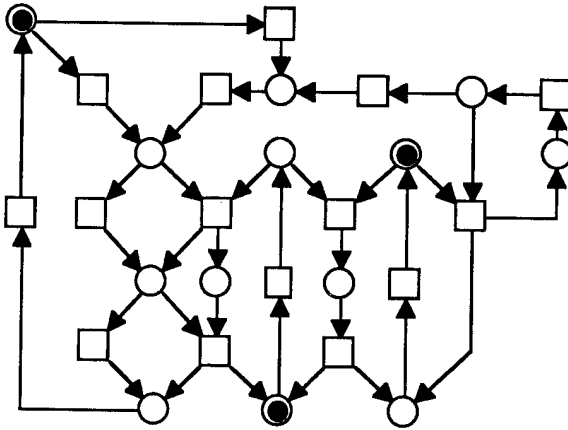


Figure 4: An SMD Petri net

**Proposition 2.5**

Let  $N$  be an SMD net. Then  $N \in \mathcal{N}(R_1, R_2)$ .

*Proof:* (sketch). We give here an outline of the synthesis procedure. Assume that  $N$  is connected (otherwise we generate its connected components separately). Let  $\mathcal{C} = \{N_1, \dots, N_r\}$  be a set of  $S$ -components of  $N$  that cover it, with the following property:

$$\forall i, 1 \leq i \leq r-1: N_{i+1} \cap \left( \bigcup_{j=1}^i N_j \right) \neq \emptyset$$

It is not difficult to see that such a set exists. The procedure synthesises first  $N_1$ , then  $N_1 \cup N_2$ ,  $N_1 \cup N_2 \cup N_3$  and so on.

Since  $N_1$  is an strongly connected  $S$ -graph, it can be obtained applying  $R_2$  to the initial net  $N_0$ .

Consider now the net  $N' = (\bigcup_{j=1}^i N_j)$ .  $N_{i+1}$  has a part in common with this net, plus one or more several "private" connected subnets. For each of these subnets, we use  $R_1$  to add a new place to  $N'$ , which is connected to  $N'$  in the same way than the corresponding subnet. Then these places are substituted by the subnets themselves using  $R_2$ . The net so obtained is  $N_{i+1} \cup (\bigcup_{j=1}^i N_j)$  ■ 2.5

We illustrate this construction by means of an example. Consider the SMD net of figure 4. The set of  $S$ -components described above is shown in figure 5. The steps of the synthesis are described in figure 6. After the first application of  $R_2$ ,  $N_1$  has been generated. We add then a place, which is expanded to yield  $N_1 \cup N_2$ . Similarly, the final net  $N = N_1 \cup N_2 \cup N_3$  is produced.



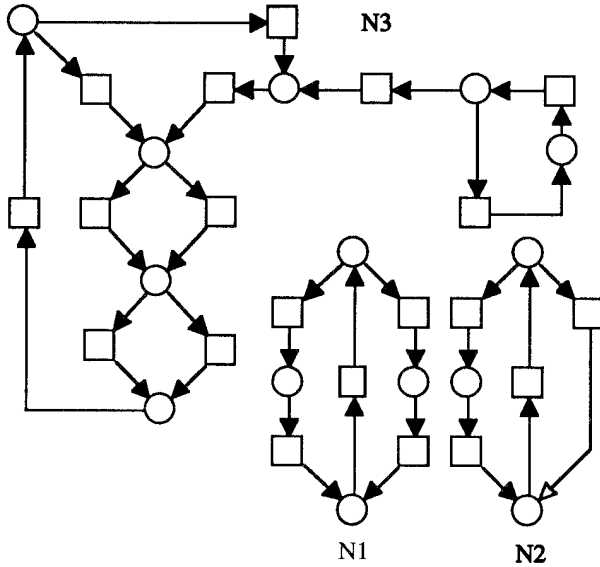


Figure 5: A set of  $S$ -components that cover the net of figure 4

**The SL&SB kit** The kit we introduce now is more interesting, because it produces only SL&SB nets (this is one of the results of [3]). Its two rules are modifications of the rules  $R_1$  and  $R_2$  of the SMD kit.

Let  $N = (S, T, F)$ ,  $\hat{N} = (\hat{S}, \hat{T}, \hat{F})$  be two nets.

**Rule 3**  $(N, \hat{N}) \in R_3$  iff:

1.  $(N, \hat{N}) \in R_1$  (i.e.  $\hat{N}$  is obtained adding a place to  $N$ )
2. The new place  $\hat{s}$  is a linear combination of places of  $S$ , i.e.  $\hat{s} = \sum_{i=1}^n \lambda_i s_i$ . ■ R3

The place  $\hat{s}$  added to the net of figure 2 satisfies the second requirement. It is not difficult to see that  $\hat{s} = s_1 + s_2$ .

**Rule 4**  $(N, \hat{N}) \in R_4$  iff the following conditions hold:

1.  $(N, \hat{N}) \in R_2$
2. The  $S$ -graph  $N' = (S', T', F') \subseteq N$  described in Rule  $R_2$  satisfies that  $\forall s' \in S', \forall$  way-out place  $s'_{out} \in S'$ : there exists an  $F'$ -path  $(s', \dots, s'_{out})$ . ■ R4

Due to condition 2, every place of  $N'$  can receive tokens (through the way-in places it is connected to), which can afterwards reach *any* of the way-out places. The idea lying behind this construction is that, if tokens are needed to fire one of the output transitions of a certain way-out place, it should be always possible for the tokens of  $N'$  to reach that place. Notice that the refinement performed with the net of figure 2 as source net and the net of the left of figure 3 as target net does not satisfy this condition: there is no  $F'$ -path from  $s'_3$  to the way-out place  $s'_2$ .

It is immediate from the definition that  $\mathcal{N}(R_3, R_4) \subseteq \mathcal{N}(R_1, R_2)$ , but the reverse is not true.

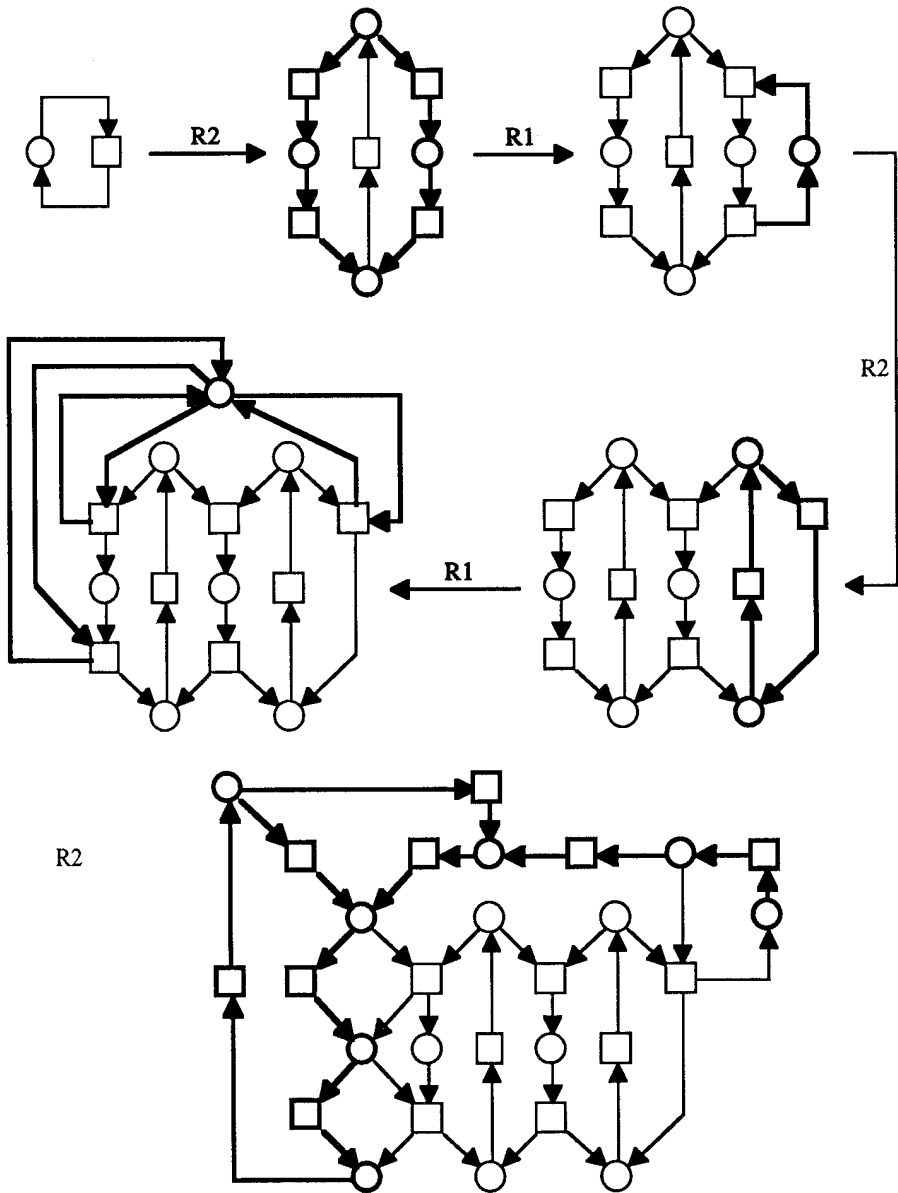


Figure 6: Synthesis of the net of figure 4

**Theorem 2.6** [3]

Let  $N$  be a net. If  $N \in \mathcal{N}(R_3, R_4)$ , then  $N$  is SL&SB.

The reader can check that the underlying net of the marked net in figure 4 belongs to  $\mathcal{N}(R_3, R_4)$  (all the applications of  $R_1$  and  $R_2$  satisfied also the conditions of  $R_3$  and  $R_4$  respectively). This net is therefore SL&SB. In fact, the net with the marking of the figure is live and bounded.

**The Free Choice kit** This kit consists of the new rule  $R_5 \subseteq R_3$ , defined below, and the rule  $R_4$ . Obviously, it produces a smaller class than the previous kit (i.e.  $\mathcal{N}(R_5, R_4) \subseteq \mathcal{N}(R_3, R_4)$ ). Nevertheless, this smaller class has a clear characterisation: namely, it is that of SL&SB Free Choice nets.

Let  $N = (S, T, F)$ ,  $\hat{N} = (\hat{S}, \hat{T}, \hat{F})$  be two nets.

**Rule 5**  $(N, \hat{N}) \in R_5$  iff:

1.  $(N, \hat{N}) \in R_3$  (i.e.  $\hat{N}$  is obtained adding a place  $\hat{s}$  that is a linear combination of places of  $N$ )
2. The new place  $\hat{s}$  satisfies  $|\hat{s}^*| = 1$ . ■ R5

We state now a property that will be used later on.

**Proposition 2.7**

$$\mathcal{N}(R_5, R_4) = \mathcal{N}(R_5, R_2).$$

*Proof:* It is obvious that  $\mathcal{N}(R_5, R_4) \subseteq \mathcal{N}(R_5, R_2)$ . To prove the other inclusion notice that, as was mentioned in remark 2.4.3,  $R_2$  is applied, except at  $N_0$ , only to the places added by the action of  $R_5$ . Since these places have exactly one output transition, the  $S$ -graphs that replace them have one single way-out place. Then condition 4 of  $R_2$  (every place is connected to at least one way-out place) is equivalent to condition 2 of  $R_4$  (every place is connected to *all* way-out places). This means that

$$(N \in \mathcal{N}(R_5, R_2) \wedge (N, \hat{N}) \in R_2) \Rightarrow (N, \hat{N}) \in R_4$$

The inclusion follows by induction on the synthesis sequences. ■ 2.7

**Theorem 2.8** [3]

Let  $N$  be an SL&SB Free Choice net. Then  $N \in \mathcal{N}(R_5, R_4)$ . ■ 2.8

Using classical theory of Free Choice nets, this result can be somewhat extended.

### Proposition 2.9 [7] [1]

The four following classes of nets are identical:

1.  $\mathcal{N}(R_5, R_4)$
2. The class of SL&SB Free Choice nets
3. The class of structurally live SMD Free Choice nets
4. The class of Free Choice nets that can be endowed with a live and bounded marking. ■ 2.9

In [5], a kit of eight rules was given for the synthesis of well behaved Bipolar Schemes. These models are included in the class of live and 1-bounded Free Choice nets, and can therefore be generated by the two-rule Free Choice kit, once the rules are slightly modified to deal with markings. Nevertheless the (modified) Free Choice kit does not guarantee 1-boundedness.

We would like to finish the section with some comments about the locality of the rules of the second and third kit. A rule is said to be *local* if, loosely speaking, the conditions for its application can be determined to hold or not by examining a small environment of a node of the graph. Local rules are easier to handle than non-local ones. In the three kits, the even-numbered rules are local, because the conditions can be checked examining only the substituted place and its input and output transitions. The odd-numbered rules are possibly easier to state, but more difficult to deal with, since they are non-local: the required linear combination could involve places of the system situated apart ones from the others. This problem was also present in the kit of [5] and, in fact, it is probably inherent to the problem. It is a widespread conjecture, though so far no proof has been given, that it is not possible to generate all and only strongly connected graphs (and therefore all and only strongly connected nets) using local rules only. Since a well known result of net theory states that live and bounded nets are strongly connected [1], we should not expect to be able to synthesise live and bounded nets using local rules only.

## 3 An algebraic characterisation of live SMD-FC nets

We prove in this section the result mentioned in the introduction: structural liveness of SMD-Free Choice nets can be characterised by means of the rank of the incidence matrix (we prove it for  $\mathcal{N}(R_5, R_4)$ , which by theorem 2.8 is the same class). More precisely, the theorem is stated as follows:

### Theorem 3.1

Let  $N = (P, T, F)$  be a net,  $C$  its incidence matrix and  $a = |F \cap (S \times T)|$  (i.e.  $a$  is the number of arcs of  $N$  leading from a place to a transition).

Then  $N \in \mathcal{N}(R_5, R_4)$  iff  $N$  is SMD and  $r(C) = n + m - a - 1$ . ■ 3.1

The 'if' part of this characterisation was conjectured by Silva [9], and the 'only if' part by Campos and Chiola [2].

The proof requires some previous definitions and lemmata. We introduce the function  $\mathcal{F} : \mathcal{N} \rightarrow \mathbb{Z}$  given by  $\mathcal{F}(N) = r(C) - (n + m - a - 1)$ . In a first step we will show that, if  $N$  is an SMD net, then  $\mathcal{F}(N) \geq 0$ . Then we will prove that  $\mathcal{F}(N) = 0$  iff  $N \in \mathcal{N}(R_5, R_4)$ .

Let us start by showing some useful relationships between  $n$ ,  $m$  and  $a$ , very easy to check.

### Proposition 3.2

Let  $N = (S, T, F)$  be a strongly connected net. Then:

1.  $a \geq n$ , and  $a = n$  if  $N$  is a Marked Graph
2.  $a \geq m$ , and  $a = m$  if  $N$  is a State Machine

*Proof:* Follows easily from the definitions of State Machine and Marked Graph. ■ 3.2

### Lemma 3.3

Let  $N$  be a strongly connected net and  $(N, \hat{N}) \in R_1$ . Then:

1.  $\mathcal{F}(\hat{N}) - \mathcal{F}(N) \geq 0$
2.  $\mathcal{F}(\hat{N}) - \mathcal{F}(N) = 0 \Leftrightarrow (N, \hat{N}) \in R_5$ .

*Proof:*  $\hat{N}$  is obtained by adding a place  $\hat{s}$  to  $N$  such that  $|\bullet \hat{s}| \neq 0 \neq |\hat{s}^\bullet|$ . This means

$$\hat{n} = n + 1 \quad \hat{m} = m \quad \hat{a} = a + |\hat{s}^\bullet| > a$$

It follows:

$$\mathcal{F}(\hat{N}) - \mathcal{F}(N) = r(\hat{C}) - r(C) - 1 + |\hat{s}^\bullet| \geq 0$$

Since  $r(\hat{C}) - r(C) \leq 1$ , the equality holds iff  $(r(\hat{C}) = r(C) \wedge |\hat{s}^\bullet| = 1)$ . But in this case  $\hat{s}$  is linear combination of places of  $S$ , and by definition  $(N, \hat{N}) \in R_5$ . ■ 3.3

### Lemma 3.4

Let  $N$  be a strongly connected net and  $(N, \hat{N}) \in R_2$ . Then  $\mathcal{F}(\hat{N}) - \mathcal{F}(N) = 0$ .

*Proof:*  $\hat{N}$  is obtained by replacing a place  $s \in S$  by an S-graph  $N' = (S', T', F')$ . We claim that  $r(\hat{C}) - r(C) = n' - 1$ . Let us prove first that if the claim is true then the result follows. It is not difficult to see that

$$\hat{n} = n + n' - 1 \quad \hat{m} = m + a' \quad \hat{a} = a + a'$$

and therefore

$$\mathcal{F}(\hat{N}) - \mathcal{F}(N) = r(\hat{C}) - r(C) - (n' - 1) - a' + a' = 0$$

Now we prove the claim.

(a)  $r(\hat{C}) - r(C) \leq n' - 1$ . This is easy, because  $\hat{C}$  has  $(n' - 1)$  rows more than  $C$  (the row corresponding to  $s$  is removed and the  $n'$  rows corresponding to  $S'$  added). It is clear that the rank cannot grow more than the difference between the number of rows of  $C$  and  $\hat{C}$ .

(b)  $r(\hat{C}) - r(C) \geq n' - 1$ . Consider the matrix  $\hat{C}_s$ , obtained by adding to  $\hat{C}$  the row of  $C$  corresponding to the place  $s$ . By the first part of remark 2.4,  $s$  is a linear combination of places of  $S'$ . Therefore  $r(\hat{C}_s) = r(\hat{C})$ .

Let  $V_1$  be a set of linearly independent rows of  $C$  with  $|V_1| = r(C)$ . Let  $V'_1$  be a set of rows of  $\hat{C}$ , corresponding to  $S'_1 \subseteq S'$ , with  $|V'_1| = n' - 1$ . We show that the vectors of  $V_1 \cup V'_1$  are linearly independent.

As  $N'$  is an S-graph,  $\exists s' \in S'_1: ((s')^* \not\subseteq S'_1 \vee {}^*(s') \not\subseteq S'_1)$  (for instance, in the net on the left of figure 3, if we remove  $s'_1$ , then  $t'_1$  and  $t'_2$  have no input place. Similarly with  $s'_2$  and  $s'_3$ ). Then  $\exists t' \in T': ({}^*t' \cup t'^*) \cap S'_1 = s'$ .

It follows that  $\hat{c}_s(s', t') \neq 0$  and  $\forall s \in S_1 \cup S'_1, s \neq s' : \hat{c}_s(s', t') = 0$ . Then clearly the row corresponding to  $s'$  is not a linear combination of the other vectors of  $V_1 \cup V'_1$ , and hence  $V_1 \cup V'_1$  is a set of linearly independent vectors. In consequence

$$r(\hat{C}) = r(\hat{C}_s) \geq |V_1 \cup V'_1| = n' - 1 + r(C)$$

and the claim is proved. ■ 3.4

These two lemmas, it is now easy to prove the following result.

### Lemma 3.5

Let  $(N_i), 0 \leq i \leq r$  be a synthesis sequence in  $\mathcal{N}(R_1, R_2)$ . Then  $\mathcal{F}$  is non-decreasing and non-negative on  $(N_i)$

$\mathcal{F}$  is non-decreasing by lemmata 3.4 and 3.3. It is non-negative because  $\mathcal{F}(N_0) = 0$  and it is non-decreasing. ■ 3.5

Now we are well prepared to prove theorem 3.1.

### Proof of theorem 3.1

( $\Rightarrow$ ): Let  $(N_i), 1 \leq i \leq r$  be a synthesis sequence of  $N$  in  $\mathcal{N}(R_3, R_5)$ . By lemmas 3.4 and 3.3

$$\forall i, 1 \leq i \leq (r - 1): \mathcal{F}(N_{i+1}) = \mathcal{F}(N_i)$$

Since  $\mathcal{F}(N_0) = 0$ , we have  $\mathcal{F}(N) = 0$ , and the result follows.

( $\Leftarrow$ ): As  $N$  is an SMD net,  $N \in \mathcal{N}(R_1, R_2)$  by proposition 2.5. Let  $(N_i), 1 \leq i \leq r$  be a synthesis sequence of  $N$  in  $\mathcal{N}(R_1, R_2)$ . Since  $\mathcal{F}$  is non-decreasing, we have

$$\forall i, 1 \leq i \leq (r - 1): \mathcal{F}(N_{i+1}) = \mathcal{F}(N_i)$$

Then, by lemma 3.3,  $N \in \mathcal{N}(R_5, R_2)$ . By proposition 2.7,  $N \in \mathcal{N}(R_5, R_4)$ . ■ 3.1

## 4 Consequences

We show in this section two results that can be derived from theorem 3.1. The first one is a new proof of Hack's duality theorem [7]. This result was proved by M. Silva [9], assuming that theorem 3.1 was true.

### Theorem 4.1

*Let  $N$  be a Free Choice net.  $N$  is SL&SB iff the reverse dual net of  $N$  is SL&SB.*

*Proof:* Assume  $N$  is SL&SB, and let  $N^{-d}$  be the reverse dual of  $N$ . We show that  $N^{-d}$  satisfies the conditions of theorem 3.1. First, it is easy to see that  $N^{-d}$  is also Free Choice. A result of [7] ensures that  $N$  is Marked Graph Decomposable. As the reverse-dual of a  $T$ -component is an  $S$ -component, it follows that  $N^{-d}$  is SMD.

Moreover, by theorem 3.1,  $\mathcal{F}(N) = n + m - 1 - a$ . Places of  $N$  correspond to transitions of  $N^{-d}$ , and viceversa. Arcs leading from places to transitions are transformed into themselves (because the arcs are reversed!). Therefore  $\mathcal{F}(N^{-d}) = m + n - 1 - a$ .

As  $N^{-d}$  satisfies all the conditions of theorem 3.1, the result holds. ■ 4.1

We can obtain from this theorem the following corollary:

### Corollary 4.2

*Let  $N$  be an FC net.  $N$  is SL&SB iff  $N$  is Marked Graph Decomposable and  $r(C) = n + m - 1 - a$ .*

*Proof:* ( $\Rightarrow$ ):  $N^{-d} = (T, P, F^{-1})$  is SL&SB by theorem 4.1. Then, by theorem 3.1,  $N^{-d}$  is SMD and  $r(-C^T) = m + n - 1 - a$ . It follows that  $N$  is SMD and  $r(C) = n + m - 1 - a$ .

( $\Leftarrow$ ):  $N^{-d}$  is SMD and satisfies the equation. Therefore, by theorem 3.1,  $N^{-d}$  is SL&SB. By theorem 4.1,  $N$  is SL&SB as well. ■ 4.2

It can be proved that a structurally live SMD-FC net is live iff all its  $S$ -components are marked at  $M_0$  [7]. Using this property, the following result is obtained in [3].

### Proposition 4.3

*Let  $(N, M_0)$  be an SMD-FC Petri net such that  $N$  is structurally live. Then  $(N, M_0)$  is live iff the following Linear Programming problem*

$$\begin{aligned} & \text{minimise} && 0^T \cdot Y \\ & \text{subject to} && Y^T \cdot C = 0 \\ & && Y^T \cdot M_0 = 0 \\ & && Y \geq 0 \end{aligned}$$

*has no solution.*

■ 4.3

Notice that the optimisation function vanishes. This means that an optimal solution exists if and only if there exists a vector  $Y \in \mathbf{Q}^n$  satisfying the constraints.

This results leads immediately to the following theorem:

## Theorem 4.4

*Liveness of marked SMD-FC nets is decidable in polynomial time.*

*Proof:* Obvious, since liveness can be decided calculating the rank of the incidence matrix and solving a Linear Programming problem, problems for which polynomial algorithms exist. ■ 4.4

## Conclusions

We have introduced and commented in this paper several kits of refinement rules for performing top-down synthesis of nets. Many results (in fact, the more difficult ones) were obtained in a previous paper. Here, free of the technical details, we have tried to give the kits a clear organisation, point out their relationships and present them in their simplest form. Our purpose was to convince the reader that small kits (in fact, containing two elements) of easy-to-describe rules suffice to produce all the nets of non-trivial subclasses enjoying desirable properties. Moreover, we wanted to show that these kits are powerful tools for deepening our knowledge on the classes of nets they generate. We hope to have achieved this by presenting a particular result: SMD-FC nets are structurally live if and only if an equation relating the rank of the incidence matrix to the number of places, transitions and arcs of the net holds. This theorem allowed us also to derive Hack's duality theorem, and to prove that liveness of SMD-FC nets is a polynomial problem.

Two interesting questions remain open, both concerning the class  $\mathcal{N}(R_3, R_4)$ : Are there simple characterisations of this class, first in terms of meaningful conditions on the structure of the nets (such as the Free Choice property is for  $\mathcal{N}(R_5, R_4)$ ), and second in purely algebraic terms? We conjecture that the answer to the first question is "yes", and to the second "no". Nevertheless, we also believe that it should be possible to find some "quasi-algebraic" characterisation of this class.

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