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# A polynomial-time algorithm to decide liveness of bounded free choice nets*

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*Abstract*

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Lautenbach (1987) described an interesting method for the linear algebraic calculation of deadlocks and traps. The method is here proved anew and its power clarified. This allows us to propose a polynomial time algorithm to decide liveness for bounded free choice nets, thus proving an enlarged version of a conjecture raised by Jones et al. (1977).

## 1. Introduction

Petri nets are a powerful tool for modelling discrete concurrent systems. One of their interesting features is the existence of a wide variety of analysis techniques. One of them is the use of so called deadlocks and traps [5, 4].

Deadlocks are sets of places which remain empty once they have lost all tokens. Traps, on the contrary, are sets of places which remain marked once they have gained ("trapped") at least one token.

The (unfortunate) name of "deadlock" derives from an easy-to-prove property [4]: when a Petri net system (or system, in the sequel) reaches a deadlock, i.e. no transition is enabled, its set of unmarked places forms a "deadlock" (with the

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meaning of the previous paragraph). Therefore if all "deadlocks" always remain marked, then the system is deadlock-free.

Deadlocks and traps become more important for subclasses of systems. The requirement that all deadlocks remain marked can be structurally achieved if all deadlocks contain initially marked traps. This condition, known as Commoner's property [8], has been proved to be necessary and sufficient for the liveness of free choice and extended free choice systems [8], non-imposed choice systems [12] and non-self-controlling systems [7]. It is also sufficient for asymmetric choice systems [9].

The practical applicability of the theory of deadlocks and traps requires efficient algorithms for their computation. The classical methods use boolean equations [14], sometimes translated into linear inequalities [1].

A new approach was studied in [11]. Deadlocks and traps were related to special $P$-semiflows of an associated net, thus opening up the possibility of applying widely used algorithms for the calculation of $P$-semiflows to the calculation of deadlocks and traps.

In [11] it was not characterized which deadlocks and traps could be obtained by the presented technique. We show here that they are the ones formed by unions of strongly connected deadlocks. The fact that not every deadlock can be obtained is—perhaps surprisingly—an advantage: this apparent limitation allows us to give here a polynomial time algorithm to decide Commoner's property, and hence liveness, for bounded free choice systems. It was conjectured in [10] that this could be achieved for conservative free choice systems. Since the latter are a subclass of the former, we also prove this conjecture.

The paper is structured as follows. In Section 2 deadlocks, traps and multisets of circuits are introduced. A summary of [11] is presented in Section 3. Section 4 gives a new proof of results somewhat stronger than those of [11]. Section 5 employs the results of Section 4 to construct a polynomial time algorithm that decides if every strongly conenected deadlock is a marked trap. Section 6 shows that this property is equivalent to Commoner's for bounded free choice systems, and therefore that the algorithm can be used to decide liveness for them in polynomial time. Basic definitions are contained in the Appendix.

## 2. Deadlocks, traps and multisets of circuits

**Definition 2.1.** Let $N=(P, T, F)$ be a net. $P'\subseteq P$ is a *deadlock* of $N$ iff $P\neq\emptyset$ and ${}^{\bullet}P'\subseteq P'^{\bullet}$. $P'\subseteq P$ is a *trap* of $N$ iff $P\neq\emptyset$ and $P'^{\bullet}\subseteq {}^{\bullet}P'$. A deadlock (trap) is *marked* iff at least one of its places is marked.

If $D$ is a deadlock and $M(D)=0$ for some marking $M$, then $M'(D)=0$ for all the markings reachable from $M$. Conversely, if $\Theta$ is a trap and $M(\Theta)>0$ for some marking $M$, then $M'(\Theta)>0$ for all the markings reachable from $M$.

**Definition 2.2.** A deadlock (trap) is *minimal* iff it does not contain a deadlock (trap) as a proper subset. A deadlock $D$ is *strongly connected* iff the subnet generated by $D \cup {}^{\bullet}D$ is strongly connected. A trap $\Theta$ is *strongly connected* iff the subnet generated by $\Theta \cup \Theta^{\bullet}$ is strongly connected.

We will make use of the two following well-known results.

**Proposition 2.3.** *The union of a set of deadlocks (traps) of a net is also a deadlock (trap).*

**Proposition 2.4.** (Hack [8]). *Minimal deadlocks and traps are strongly connected.*

**Definition 2.5.** Let $N = (P, T, F)$ be a net. A *multiset of circuits* is a collection of circuits of $N$ that may contain several copies of an element. Given a multiset of circuits $L$ and $y \in P \cup T \cup F$, $L(y)$ denotes the number of circuits of $L$ that contain $y$. If $L(y) > 0$, $L$ *covers* $y$. The *support* of $L$, denoted by $\|L\|$, is the set of places that $L$ covers.

In the sequel only multisets of elementary circuits will be considered. We will drop the adjective "elementary" when referring to them.

## 3. Deadlocks and traps can be calculated as *P*-semiflows of an associated net

Let us summarize the results of [11], though for a complete description the reference should be consulted. The statement of the title above is proved in two stages, that correspond to the two parts of this section.

### *3.1. Deadlocks (traps) are related to graph constructions called d-multisets of circuits (θ-multisets of circuits)*

**Definition 3.1.** Let $N = (P, T, F)$ be a net and $L$ a multiset of circuits of $n$. $L$ is a

- *d-multiset of circuits* iff $L \neq \emptyset$ and $\forall p \in P \exists k_p \in \mathbb{N}$ such that $\forall t \in {}^{\bullet}p$: $L[(t, p)] = k_p$,
- *θ-multiset of circuits* iff $L \neq \emptyset$ and $\forall p \in P \exists k_p \in \mathbb{N}$ such that $\forall t \in p^{\bullet}$: $L[(p, t)] = k_p$.

That is, the same number of circuits $k_p \geqslant 0$ passes through all the input arcs for $d$-multisets, output arcs for $\theta$-multisets, of a place $p$. $d$-multiset and $\theta$-multiset of circuits will be abbreviated to $d$-mc and $\theta$-mc, respectively.

The reader can easily check from the definition that the union of two $d$-mcs is also a $d$-mc, and so is the multiplication of a $d$-mc by a positive integer (analogously for $\theta$-mcs). We introduce now minimal $d$-mcs and $\theta$-mcs.

**Definition 3.2.** A $d$-mc is *minimal* iff the following two conditions hold:
(a) its support does not contain the support of a $d$-mc as a proper subset,
(b) the g.c.d. of the numbers $k_p$ is 1.
The corresponding definition for $\theta$-mcs is analogous.

Notice that the notation of [11] differs from ours: there, $d$-mcs and $\theta$-mcs are called $D$-systems and $T$-systems of circuits respectively.

The relationship between $d$-mcs and deadlocks, and $\theta$-mcs and traps is given by the following theorem.

**Theorem 3.3** (Lautenbach [11]). *Let $N=(P, T, F)$ be a pure and strongly connected net.*
(a) *If $L$ is a d-mc ($\theta$-mc), then $\|L\|$ is a deadlock (trap).*
(b) *If $P'$ is a minimal deadlock (trap), then there exists a minimal d-mc ($\theta$-mc) $L$ such that $\|L\|=P'$.*

The proof of (a) follows easily from the definitions, while (b) is non-trivial. In the next section, a slightly stronger theorem will be proved. In particular, it will be shown that Theorem 3.3 also holds for non-pure nets. That is why we illustrate the theorem with the example of Fig. 1, which is a non-pure net. The net contains the following circuits:

$$\Gamma_1=(t_1, p_1, t_1) \qquad \Gamma_5=(t_1, p_2, t_3, p_3, t_2, p_5, t_1)$$
$$\Gamma_2=(t_1, p_1, t_2, p_5, t_1) \qquad \Gamma_6=(t_1, p_2, t_3, p_5, t_1)$$
$$\Gamma_3=(t_1, p_1, t_3, p_3, t_2, p_5, t_1) \qquad \Gamma_7=(t_3, p_3, t_4, p_4, t_3)$$
$$\Gamma_4=(t_1, p_1, t_3, p_5, t_1)$$

The multiset $L=\{\Gamma_2, \Gamma_4\}$ is a $d$-mc (notice that neither $\{\Gamma_2\}$ nor $\{\Gamma_4\}$ are $d$-mcs, because they cover only one of the two input arcs of $p_5$). We have $\|L\|=\{p_1, p_5\}$,

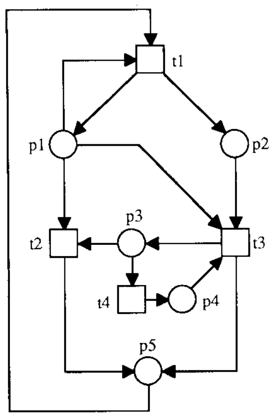

Fig. 1.

which is a deadlock of $N$. Notice nevertheless that, although $L$ is a minimal $d$-mc, $\|L\|$ is *not* a minimal deadlock ($\{p_1\}$ is a deadlock as well). Only the converse is true: for instance, $\{p_2, p_3, p_5\}$ is a minimal deadlock of $N$ and is the support of the minimal $d$-mc $\{\Gamma_5, \Gamma_6\}$.

We have then seen that some deadlocks can be calculated from the support of $d$-mcs. In the second part of the section an outline of the technique proposed by Lautenbach to calculate these supports is given. Before that, we introduce the concept of shared node and state a well-known result, on which the technique is based, which relates $P$-semiflows of $T$-graphs to multisets of circuits.

**Definition 3.4.** Let $N=(P, T, F)$. A place $p$ is *input shared* iff $|{}^{\bullet}p|>1$. $p$ is *output shared* iff $|p^{\bullet}|>1$. The set of input shared and output shared nodes of $N$ are denoted by IS and OS, respectively. $p$ is *shared* iff it is input shared or output shared.

**Lemma 3.1** (Lautenbach [11]). *Let $N=(P, T, F)$ be a T-graph. Then*:

(a) *$X$ is a minimal P-semiflow of $N$ iff there exists a circuit $\Gamma$ of $N$ such that $\|X\|=\|\Gamma\|$*,

(b) *$X$ is a P-semiflow of $N$ iff there exists a multiset $L$ of circuits of $N$ such that $\forall p \in P$: $X(p)=L(p)$.*

### *3.2. The supports of d-mcs and θ-mcs can be calculated as special P-semiflows of associated nets*

Let $N=(P, T, F)$ be a net. The calculation can be divided into three steps.

*Step 1. Expansion of the net*

A net $\bar{N}=(\bar{P}, \bar{T}, \bar{F})$ is constructed through an expansion of $N$. This expansion modifies only shared places and is graphically described in Fig. 2. The expansion

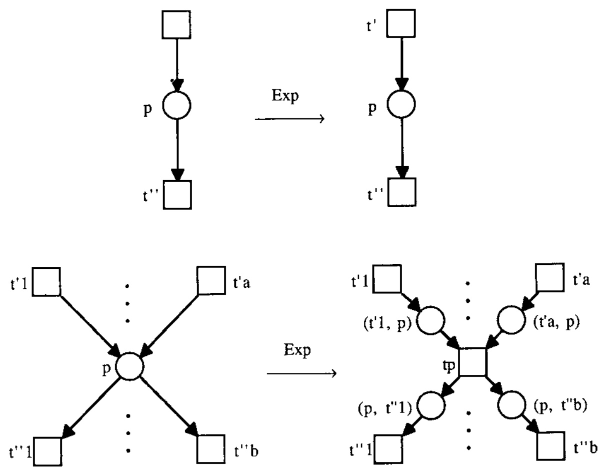


Fig. 2. Expansion rule given in Lautenbach [11].

does not remove any transition of the net $N$. Moreover, $\bar{N}$ satifies $\forall \bar{p} \in \bar{P}$: $|{}^{\bullet}p| \leqslant 1$ and $|p^{\bullet}| \leqslant 1$. If $\mathrm{Exp}(p)$ denotes the set of places of $\bar{N}$ produced by the expansion of $p$, then

- $\mathrm{Exp}(p) = \{p\}$ if $p$ is not shared (Fig. 2(a)).
- $\mathrm{Exp}(p) = \{(t'_1, p), \ldots, (t'_a, p), (p, t''_1), \ldots, (p, t''_b)\}$ if $p$ is shared (Fig. 2(b)).

Notice that the new places of $\bar{N}$ correspond to arcs of the net $N$, and we label them accordingly.

It is shown in [11] that some $P$-semiflows of this expanded net correspond to the $d$-mcs and $\theta$-mcs of $N$. This subset of $P$-semiflows can be characterized by adding some constraints to the $P$-semiflow defining equation system $X \cdot \bar{C} = 0$, where $\bar{C}$ is the incidence matrix of $\bar{N}$. That is the purpose of the second step.

*Step 2. Addition of constraints to the equation system* $X \cdot \bar{C} = 0$

The following constraints are added.

Case of deadlocks:

$$\forall p \in \mathrm{IS}\ \exists k_p \in \mathbb{N} \text{ such that } \forall t \in {}^{\bullet}p\colon X[(t, p)] = k_p \tag{3.1}$$

Case of traps:

$$\forall p \in \mathrm{OS}\ \exists k_p \in \mathbb{N} \text{ such that } \forall t \in p^{\bullet}\colon X[(p, t)] = k_p \tag{3.2}$$

Intuitively, these constraints select the multisets of circuits that pass the same number of times by all input (output) arcs of each input (output) shared place. For calculations it is better to express (3.1) and (3.2) as equations, removing the constant $k_p$. Let $p \in \mathrm{IS}$ and ${}^{\bullet}p = \{t_1, \ldots, t_a\}$. Then, for $p$ the condition (3.1) is equivalent to

$$\begin{aligned} -X[(t_1, p)] + X[(t_2, p)] &= 0 \\ -X[(t_2, p)] + X[(t_3, p)] &= 0 \\ &\vdots \\ -X[(t_{a-1}, p)] + X[(t_a, p)] &= 0. \end{aligned} \tag{3.3}$$

and similarly for (3.2).

In the sequel we denote the augmented system (system $X \cdot \bar{C} = 0$ plus constraints) by

$$X \cdot \bar{C}_d = 0 \quad \text{(deadlocks)} \tag{3.4}$$

$$X \cdot \bar{C}_\theta = 0 \quad \text{(traps)} \tag{3.5}$$

where $\bar{C}_d$ and $\bar{C}_\theta$ are $\bar{C}$ enlarged with the respective constraints. We can interpret $\bar{C}_d$ and $\bar{C}_\theta$ as the incidence matrices of two nets $\bar{N}_d$ and $\bar{N}_\theta$, respectively. The reader can check that equations (3.3) correspond to new transitions $td_{i-1,i}$ $1 < i \leqslant a$, having only $(t'_{i-1}, p)$ as input place and only $(t'_i, p)$ as output place (see Fig. 3).

The net of Fig. 4 is the expansion of the net of Fig. 1, to which the constraints for the case of deadlocks have been added (in fact, the only new transition that has to be added is $td_{2,3}$).

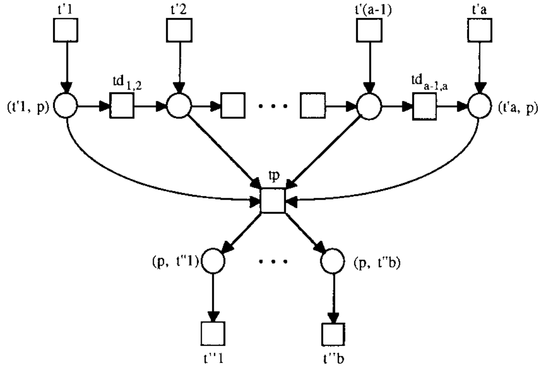


Fig. 3. Constraints seen as transitions (case of deadlocks).

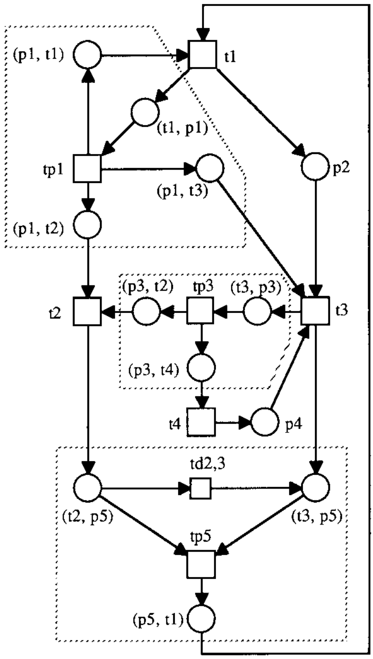


Fig. 4. Expansion of the net of Fig. 1, with the constraints (3.3) seen as transitions. The expansion of the shared places is indicated by the dashed boxes.

The incidence matrix corresponding to this net is shown in Table 1.

Finally, we obtain from the $P$-semiflows of the net $\bar{N}_d$ ($\bar{N}_\theta$) the supports of the $d$-mcs ($\theta$-mcs) of the original net.

*Step 3. Computation of the supports of d-mcs or θ-mcs*

A subset $\|X\|_N \subseteq P$ is associated with each solution $X$ of (3.4) ((3.5) for traps) in the following way: $p \in \|X\|_N$ iff at least one of the places of $\mathrm{Exp}(p)$ belongs to $\|X\|$.

Table 1
Incidence matrix corresponding to the net of Fig. 4

| | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $tp_1$ | $tp_3$ | $tp_5$ | $td_{2,3}$ |
|---|---|---|---|---|---|---|---|---|
| $(t_1, p_1)$ | 1 | 0 | 0 | 0 | −1 | 0 | 0 | 0 |
| $(p_1, t_1)$ | −1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $(p_1, t_2)$ | 0 | −1 | 0 | 0 | 1 | 0 | 0 | 0 |
| $(p_1, t_3)$ | 0 | 0 | −1 | 0 | 1 | 0 | 0 | 0 |
| $p_2$ | 1 | 0 | −1 | 0 | 0 | 0 | 0 | 0 |
| $(t_3, p_3)$ | 0 | 0 | 1 | 0 | 0 | −1 | 0 | 0 |
| $(p_3, t_2)$ | 0 | −1 | 0 | 0 | 0 | 1 | 0 | 0 |
| $(p_3, t_4)$ | 0 | 0 | 0 | −1 | 0 | 1 | 0 | 0 |
| $p_4$ | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 |
| $(t_2, p_5)$ | 0 | 1 | 0 | 0 | 0 | 0 | −1 | −1 |
| $(t_3, p_5)$ | 0 | 0 | 1 | 0 | 0 | 0 | −1 | 1 |
| $(p_5, t_1)$ | −1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

The vectors $X_1 = (2\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 2)$ and $X_2 = (0\ 0\ 0\ 0\ 2\ 1\ 1\ 0\ 0\ 1\ 1\ 2)$ are solutions of (3.4) for the net of Fig. 4. We obtain:

$$\|X_1\| = \{(t_1, p_1), (p_1, t_2), (p_1, t_3), (t_2, p_5), (t_3, p_5), (p_5, t_1)\},$$

$$\|X_1\|_N = \{p_1, p_5\},$$

$$\|X_2\| = \{p_2, (t_3, p_3), (p_3, t_2), (t_2, p_5), (t_3, p_5), (p_5, t_1)\},$$

$$\|X_2\|_N = \{p_2, p_3, p_5\}.$$

In [11] the following theorem is easily derived from Lemma 3.5.

**Theorem 3.6** (Lautenbach [11]). *Let $N=(P, T, F)$ be a net and $P' \subseteq P$. $P'$ is the support of a d-mc ($\theta$-mc) of $N$ iff there exists a P-semiflow $X$ of $\bar{N}_d$ ($\bar{N}_\theta$) such that $\|X\|_N = P'$.*

In the examples, $\|X_1\|_N$ and $\|X_2\|_N$ are the supports of the d-mcs $\{\Gamma_2, \Gamma_4\}$ and $\{\Gamma_5, \Gamma_6\}$ respectively.

Theorem 3.6 shows that, after Step 3, the set of all the supports of d-mcs or $\theta$-mcs of $N$ has been obtained. By Theorem 3.3(a), this is a set of deadlocks (traps) of $N$. Moreover, by Theorem 3.3(b), the set contains all the minimal deadlocks (traps). Nevertheless, it does not contain all deadlocks of $N$. Consider the underlying net $N$ of the system of Fig. 5.

Since the net contains no shared places, the expanded net $\bar{N}$ is $N$ itself, and we have also $\bar{N}_d = N$. The set $\{p_1, p_2, p_3\}$ is a deadlock. Nevertheless, there is no $P$-semiflow with that support. It will be shown in Section 4 that the deadlocks that can be obtained by means of $P$-semiflows are exactly those that are union of strongly connected deadlocks.

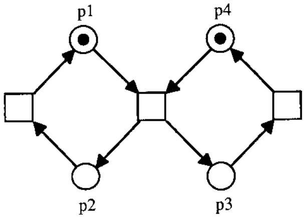


Fig. 5. No $P$-semiflow has $D=\{p_1, p_2, p_3\}$ as support.

## 4. New proof of correctness for Lautenbach's technique

Although the approach of [11] is of high interest, we consider that Theorem 3.3 can be improved in two ways. First, the theorem does not characterize the set of deadlocks and traps that can be calculated using $P$-semiflows: we only know that this set includes all minimal deadlocks but not all deadlocks. Second, the proof given in [11] holds only for pure and strongly connected nets.

We state in this section a theorem (Theorem 4.2) slightly stronger than Theorem 3.3. It holds for any net in which every place has at least one input transition. The theorem clarifies that the technique allows the calculation of all the unions of strongly connected deadlocks and traps, and only of them. This slight improvement will turn out to be the key for the results of Sections 5 and 6. The basic idea of our approach is contained in Theorem 4.1. Theorem 4.2 states the final result.

**Theorem 4.1.** *Let $N=(P, T, F)$ be a strongly connected net with $T\neq\emptyset$. Then $N$ can be covered by a d-mc.*

**Proof** (by induction on the number $k$ of input shared places, $k=|\mathrm{IS}|$).

*Base.* $k=0$. $N$ can be covered by circuits (it is strongly connected), and this covering is a $d$-mc.

*Step.* Assume that every strongly connected net with $k$ or less shared places can be covered by a $d$-mc. Let $N=(P, T, F)$ be a strongly connected net with $|\mathrm{IS}|=k+1$. Choose $p\in\mathrm{IS}$. Let ${}^{\bullet}p=\{t_1, \ldots, t_a\}$. We construct now for each $t_i$, $1\leq i\leq a$, a partial subnet $N_i=(P_i, T_i, F_i)$ as follows. Let $\Sigma_i$ be the set of (not necessarily elementary) paths $(x_1, \ldots, x_r)$ of $N$ such that:

(i) $x_r=p$,

(ii) $\forall j$, $1<j\leq r$: $x_j=p \Rightarrow x_{j-1}=t_i$,

i.e. paths that "enter $p$" only through $t_i$.

Let $N_i$ be the net covered by all the paths of $\Sigma_i$ (see Fig. 6 for an example). We make the following claims about $N_i$:

(1) $N_i$ is strongly connected.

*Proof of claim* 1. Take $x\in P_i\cup T_i$. $x$ can be connected to $p$ in $N_i$ through a path $\Pi_1$ of $\Sigma$ by construction. Let us see now that $p$ can be connected to $x$ in $N_i$. As $N$ is strongly connected, there exists an elementary path $\Pi_2$ of $N$ from $p$ to $x$. Then

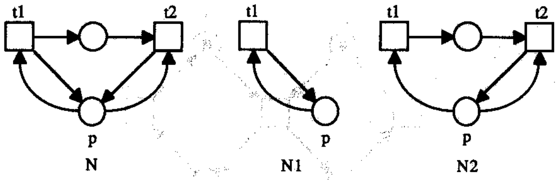


Fig. 6. The two nets on the right are the ones obtained from the net on the left for $t_1, t_2$; notice that $N_2$ is not a subnet but a partial subnet of $N$.

$\Pi_2;\Pi_1$ (the concatenation of $\Pi_2$ and $\Pi_1$) is also a path of $\Sigma_i$, and therefore $\Pi_2$ is in $N_i$.

(2) If $p' \neq p$ and $p' \in P_i$, then $T_i$ contains ${}^{\bullet}p'$.

*Proof of claim* 2. If $(p', \ldots, p)$ is a path of $\Sigma_i$, so is $(t, p', \ldots, p)$ for all $t \in {}^{\bullet}p'$.

(3) $\bigcup_{i=1}^{a} N_i = N$.

*Proof of claim* 3. Obvious.

(4) $\forall i$, $1 < i \leq a$: $N_i$ has $k$ or less input shared places.

*Proof of claim* 4. Take just into account that $T_i \cap {}^{\bullet}p = \{t_i\}$ and therefore $p$ is not an input shared place of $N_i$.

Using (4) and the induction hypothesis, we conclude that every $N_i$ can be covered by a $d$-mc $L_i$ of $N_i$. Let $\Lambda(p)$ be the least common multiple of the numbers $L_i(p)$, $1 \leq i \leq a$. Consider the multiset of circuits

$$L = \sum_{i=1}^{a} (\Lambda(p)/L_i(p))L_i$$

Since the union and the multiplication by a positive constant are internal operations on $d$-mcs, $L$ is a $d$-mc. Moreover, since $L_i$ covers $N_i$, $L$ covers $N$. $\square$

**Theorem 4.2.** *Let $N = (P, T, F)$ be a net such that every place has at least one input transition and let $P' \subseteq P$. Then the three following statements are equivalent.*

(a) *$P'$ is the union of a set of pairwise disjoint strongly connected deadlocks of $N$.*

(b) *There exists a d-mc $L$ of $N$ with $P'$ as support.*

(c) *There exists a P-semiflow $X$ of $\bar{N}_d$ such that $\|X\|_N = P'$.*

**Proof.** (a$\Rightarrow$b) Let $\{D_1, \ldots, D_a\}$ be the set of strongly connected deadlocks whose union yields $P'$. Since every $D_i$, $1 \leq i \leq a$, is a strongly connected deadlock, the subnet $N_i$ generated by $D_i \cup {}^{\bullet}D_i$ is strongly connected (Definition 2.2), and contains at least one transition because every place has at least one input transition. By Theorem 4.1, $N_i$ can be covered by a $d$-mc $L_i$ of $N_i$. Since $p \in P_i$ implies ${}^{\bullet}p \subseteq T_i$, we have that $L_i$ is also a $d$-mc of $N$. Then the union of the $L_i$ for $1 \leq i \leq a$ is a $d$-mc of $N$ with $P'$ as support.

(b$\Rightarrow$a) Let $N_i = (P_i, T_i, F_i)$ be a connected component of the partial subnet of $N$ covered by $L$. $N_i$ is strongly connected, because it is covered by circuits of $L$. Moreover, $p \in P_i$ implies ${}^{\bullet}p \subseteq T_i$ because of the $d$-mc property. Then $P_i$ is a strongly connected deadlock of $N$ (the subnet generated by $P_i \cup {}^{\bullet}P_i$ is just $N_i$).

(b$\Leftrightarrow$c) See Theorem 3.6. $\square$

In the net of Fig. 1, $\{p_1\}$ and $\{p_3, p_4\}$ are both strongly connected deadlocks of $N$. $\{\Gamma_1, \Gamma_7\}$ is a $d$-mc with $\{p_1, p_3, p_4\}$ as support. The semiflow $X = (1\,1\,0\,0\,0\,1\,0\,1\,1\,0\,0\,0)$ of the net $\bar{N}$ (Fig. 4) satisfies $\|X\|_N = \{p_1, p_2, p_4\}$.

Part (b) of Theorem 3.3 follows now as a corollary.

**Corollary 4.3.** *Let $N = (P, T, F)$ be a net and $D$ a minimal deadlock of $N$. Then:*

(a) *there exists a minimal d-mc $L$ such that $\|L\| = D$,*

(b) *there exists a minimal P-semiflow $X$ of $\bar{N}_d$ such that $\|X\|_N = D$.*

**Proof.** (a) By Proposition 2.4, $D$ is strongly connected. By Theorem 4.2, there exists a $d$-mc $L$ such that $\|L\| = D$. Assume that there exists another $d$-mc $L' \neq L$ satisfying $\|L'\| \subseteq \|L\|$. By Theorem 4.2 again, $\|L'\|$ is a strongly connected deadlock of $N$, what contradicts the minimality of $P'$. Therefore $L$ satisfies condition (a) of minimality (Definition 3.2). Let now $k$ be the g.c.d. of the numbers $k_p$. If $k > 1$, let $L'' = (1/k)L$. Then $\|L''\| = \|L\|$ and the g.c.d. of the numbers $k''_p$ is 1, which implies that $L''$ is minimal.

(b) Analogous to (a). $\square$

The minimal deadlocks of the net of Fig. 1 are $\{p_1\}$, $\{p_3, p_4\}$ and $\{p_2, p_3, p_5\}$. Corollary 4.3 ensures that the corresponding $d$-mcs can be chosen minimal: $\{\Gamma_1\}$, $\{\Gamma_7\}$ and $\{\Gamma_5, \Gamma_6\}$ satisfy this requirement.

Summarizing, Theorem 3.6 showed that the algorithm outlined in Section 3.2 calculates the supports of all the $d$-mcs of a net. Theorem 4.2 proves that these supports are all the unions of pairwise disjoint strongly connected deadlocks.

## 5. A polynomial time algorithm to decide if every strongly connected deadlock of a system is a marked trap

Using the results of Section 4, we prove now that every strongly connected deadlock of a system is a marked trap iff at least one of a set of systems of linear inequalities has a nonzero solution. The number and size of the systems will be polynomial functions on the number of arcs of the net. We shall make use of the following technical lemma.

**Lemma 5.1.** *Let $N' = (P', T', F')$ be a subnet of $N = (P, T, F)$, obtained by removing places from $N$, together with their input and output arcs, and $Q \subseteq P'$. Then $Q$ is a deadlock of $N'$ iff it is a deadlock of $N$. Moreover, $Q$ is strongly connected in $N'$ iff it is strongly connected in $N$.*

**Proof.** Since $T' = T$, we have ${}^\bullet Q \cap T' = {}^\bullet Q$ and $Q^\bullet \cap T' = Q^\bullet$. Hence, ${}^\bullet Q \subseteq Q^\bullet$ iff ${}^\bullet Q \cap T' \subseteq Q^\bullet \cap T'$. Moreover, $Q \cup {}^\bullet Q = Q \cup ({}^\bullet Q \cap T')$, and therefore the subnets generated by $Q \cup {}^\bullet Q$ and $Q \cup ({}^\bullet Q \cap T')$ coincide. □

**Theorem 5.2.** *Let $(N, M_0)$ be a system where $N = (P, T, F)$. It can be decided in polynomial time in $|F|$ if every strongly connected deadlock of $N$ is a marked trap.*

**Proof.** We can assume without loss of generality that $N$ contains neither isolated places nor isolated transitions. We consider first the case in which $N$ contains a place with no input transitions. Then this place is a strongly connected deadlock of $N$ but not a trap. Since these places can be detected in linear time on the number of places, we are done.

Assume then that every place has at least one input transition. The algorithm we present has the following logical form:

> **if** every strongly connected deadlock is marked
> **and** every strongly connected deadlock is a trap
> **then** Answer = Yes
> **else** Answer = No

We show first how to calculate the logical values of the conditions by means of systems of linear inequalities. Then we make an estimation of the cost of the algorithm.

*Checking if every strongly connected deadlock is marked*

Let $N' = (P', T', F')$ be the subnet obtained removing from $N$ all the places $p$ such that $M_0(p) > 0$, together with their input and output arcs. By Lemma 5.1, $D$ is an unmarked strongly connected deadlock of $N$ iff it is a strongly connected deadlock of $N'$ (because $D \subseteq P'$). Hence, it suffices to check if $N'$ contains a strongly connected deadlock. By Theorem 4.2 (equivalence of (a) and (c)), $N'$ contains a strongly connected deadlock iff there exists a $P$-semiflow of $\bar{N}'_d$. This can be decided checking if the following system S1 of inequalities has a nonzero integer solution.

$$\text{S1} \qquad X \cdot \bar{C}'_d = 0 \qquad X \geqslant 0$$

*Checking if every strongly connected deadlock is a trap*

For each $t \in \mathrm{OS}^\bullet$, consider the subnet $N^t = (P^t, T^t, F^t)$ obtained by removing from $N$ the places of $t^\bullet$ together with their input and output arcs.

We claim that $N$ contains a strongly connected deadlock $D$ that is not a trap iff $D$ is a strongly connected deadlock of $N^t$ for some transition $t \in \mathrm{OS}^\bullet$ satisfying ${}^\bullet t \cap D \neq \emptyset$.

*Proof of the claim.* ($\Rightarrow$) Since $D$ is not a trap, there exists $t \in D^\bullet \setminus {}^\bullet D$. Let $p \in {}^\bullet t \cap D$. Since $D$ is strongly connected, there exists $t' \in p^\bullet$ such that $t' \in {}^\bullet D$. Hence $t \neq t'$, which together with $t, t' \in p^\bullet$ implies $p \in \mathrm{OS}$ and $t \in \mathrm{OS}^\bullet$. Moreover, $D \subseteq P^t$. By Lemma 5.1, $D$ is a strongly connected deadlock of $N^t$.

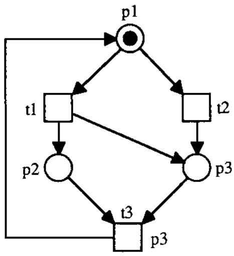


Fig. 7. Illustration of the proof of Theorem 5.2.

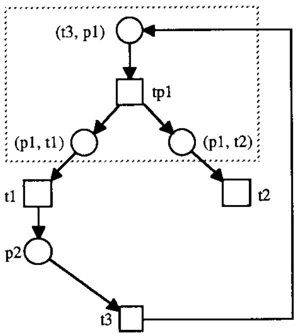


Fig. 8. Expansion of the net $N^{t2}$ and incidence matrix.

($\Leftarrow$) By Lemma 5.1, $D$ is a strongly connected deadlock of $N$. We have ${}^{\bullet}t \cap D \neq \emptyset$ by hypothesis and $t^{\bullet} \cap D = \emptyset$ by construction of $N^{t}$. It follows that $t \in D^{\bullet} \setminus {}^{\bullet}D$.

In the system of Fig. 7, $\{p_1, p_2\}$ is a strongly connected deadlock that is not a trap. To obtain $N^{t2}$ we remove $p_3$ together with its input and output arcs. $\{p_1, p_2\}$ is a strongly connected deadlock of $N^{t2}$ such that ${}^{\bullet}t_2 \cap \{p_1, p_2\} \neq \emptyset$.

We show now that $N^{t}$ contains a strongly connected deadlock $D$ satisfying ${}^{\bullet}t \cap D \neq \emptyset$ iff at least one of the systems of inequalities of a certain set has an integer solution. Solving the set of systems corresponding to all the transitions of $\mathrm{OS}^{\bullet}$, we can deduce if $N$ contains a strongly connected deadlock that is not a trap.

Let $\bar{C}^{t}_{d}$, be the incidence matrix of the expansion of $N^{t}$ with the constraints (3.3). The expansion of $N^{t2}$ is shown in Fig. 8. Its corresponding incidence matrix is depicted in Table 2.

Table 2
Incidence matrix corresponding to the net of Fig. 8

| | $t_1$ | $t_2$ | $t_3$ | $tp_1$ |
|---|---|---|---|---|
| $(t_3, p_1)$ | 0 | 0 | 1 | $-1$ |
| $(p_1, t_1)$ | $-1$ | 0 | 0 | 1 |
| $(p_1, t_2)$ | 0 | $-1$ | 0 | 1 |
| $p_2$ | 1 | 0 | $-1$ | 0 |

The set of systems of inequalities corresponding to $t$ contains one element for each place $p \in \mathrm{OS} \cap {}^{\bullet}t$. This element is called $S2(p, t)$ and has the following form:

$$\begin{array}{lll} S2(p, t) & X \cdot \bar{C}_d^t = 0, & \\ & X \geqslant 0 & (X \text{ is a } P\text{-semiflow of } N^t), \\ & X[(t', p)] > 0 & (\|X\| \text{ contains the place } (t', p)), \end{array}$$

where $(t', p)$ is arbitrarily selected among the input places of $tp$ in $\bar{N}_d^t$.

In our example, the set contains one single element $S2(p_1, t_2)$, and $(t' p) = (t_3, p_1)$.

Assume now that $N^t$ contains a strongly connected deadlock $D$ such that there exists $p \in {}^{\bullet}t \cap D$. We show that $S2(p, t)$ has a solution. By Theorem 4.2 (equivalence of (a) and (c)), there exists a $P$-semiflow $X$ of $\bar{N}_d^t$ such that $\|X\|_{N^t} = D$. Since $X$ is a $P$-semiflow, it satisfies the two first equations of $S2(p, t)$. By Theorem 4.2 (equivalence of (a) and (b)), there exists a $d$-mc $L$ such that $\|L\| = \|X\|_{N^t}$ and $p \in \|L\|$. By the definition of $d$-mc, $L$ covers all the input arcs of $p$ in $N$. This implies that $\|X\|$ contains all the places of the form $(x, p)$, in particular $(t', p)$. Hence, $X$ satisfies also the third equation, and is a solution of $S2(p, t)$.

In our example, $X = (1\,1\,0\,1)$, which covers $(t_3, p_1)$.

Assume now that for every strongly connected deadlock $D$ of $N^t$, ${}^{\bullet}t \cap D \neq \emptyset$. By Theorem 4.2 and the definition of Exp, every $P$-semiflow $X$ of $\bar{N}_d^t$ satisfies $\|X\| \cap \mathrm{Exp}(p) = \emptyset$, and therefore $X[(t', p)] = 0$.

*Cost of the algorithm*

Since we are interested in the solutions of the systems of inequalities, it would appear that we have to use integer linear programming in order to solve them. Nevertheless, since they are all homogeneous, they have a (nonzero) integer solution iff they have a (nonzero) rational one.

Systems of linear equations can be solved on the nonnegative orthant in polynomial time on the size of the system. Many different algorithms have been proposed in the literature. Since our purpose is to obtain an estimation, we shall consider a particular one, presented in [6]. Let $n$ be the number of variables of the system, $m$ its number of equations and $L = nm + \lfloor \log_2 |G| \rfloor + 1$ its size, where $G$ is the product of the nonzero coefficients. The algorithm decides in at most $O(n^2 m^2 L)$ operations if the system has a (nonzero) solution. Since, in our case, all the nonzero coefficients are 1 or $-1$, $O(n^2 m^2 L) = O(n^3 m^3)$.

In the set of equations S1, $n$ and $m$ are the numbers of places and transitions, respectively, of the net $\bar{N}'_d$. Since we assume that there exist no isolated places nor transitions, both $n$ and $m$ are $O(|F|)$, where $F$ is the number of arcs of the original net. In the sets $S2(p, t)$, $n$ and $m$ are the number of places and transitions of the net $\bar{N}_d^t$ and, once again, they are $O(|F|)$. Hence, we can decide that one of the equation sets has no solution in $O(|F|^6)$. A set S2 has to be solved for each place $p \in \mathrm{OS}$ and each transition $t \in p^{\bullet}$. The number of equation sets is thus $\sum_{p \in \mathrm{OS}} |p^{\bullet}| \leqslant |P||T|$, and the cost of the algorithm $O(|F|^6 |P||T|)$. Since both $|P|$ and $|T|$ are $O(|F|)$ as well, the cost is also $O(|F|^8)$. □

## 6. Application of the algorithm to deciding liveness of bounded free choice nets

Commoner's property (defined below) is involved in results about liveness of many subclasses of nets, as mentioned in the introduction. The practical applicability of the theory requires efficient algorithms in order to decide if a given net satisfies the property or not. This problem was approached in [13], where a fast polynomial time algorithm based on resolution of Horn clauses was presented, which decided if every deadlock of the net is a trap. Unfortunately, there exist even live $T$-systems (net systems whose underlying net is a $T$-graph) that do not satisfy this property. An example is given in Fig. 5: the system is live, but the deadlock $\{p_1, p_2, p_3\}$ is not a trap.

There is however an upper bound (assuming that P$\neq$NP) on how far a polynomial time algorithm can go: to decide if a free choice system is non-live is an NP-complete problem [10]. Since this problem is equivalent to deciding that Commoner's property does not hold, it is unlikely that a polynomial time algorithm exists to decide Commoner's property for the class of free choice systems.

Our problem is to find such an algorithm for an interesting subclass larger than $T$-systems. It was conjectured in [10] that this algorithm existed for conservative free choice systems. We show in this section that the conjecture is true even for bounded free choice systems. The proof is carried out by showing that a system in this subclass is live if and only if every strongly connected deadlock is a marked trap. We use then the algorithm of Section 5.

**Definition 6.1.** A system $(N, M_0)$ satisfies Commoner's property iff every minimal deadlock of $N$ contains a marked trap.

The following theorem shows the relationship between Commoner's property and free choice systems.

**Theorem 6.2** (Hack [8], Best and Desel [3]).
(a) *A free choice system is live iff it satisfies Commoner's property.*
(b) *In a live and bounded free choice system, every minimal deadlock is a marked trap.*

We can easily derive the following corollary.

**Corollary 6.3.** *A bounded free choice system is live iff every minimal deadlock is a marked trap.*

**Proof.** ($\Rightarrow$) Theorem 6.2(b).
($\Leftarrow$) If a minimal deadlock is a marked trap, then it contains a marked trap. Hence, Commoner's property is satisfied and, by Theorem 6.2(a), the system is live. □

The rest of the section is devoted to proving that Corollary 6.3 remains true if we substitute "minimal" by "strongly connected". We need to have a closer look at the minimal deadlocks of free choice nets.

**Theorem 6.4.** *Let $N=(P,T,F)$ be a free choice net, $D\subseteq P$ a deadlock of $N$ and $N_D=(P_D,T_D,F_D)$ the subnet of $N$ generated by $D\cup {}^{\bullet}D$. $D$ is minimal iff it is strongly connected and for every transition $t\in T_D$: $|{}^{\bullet}t\cap D|\leqslant 1$.*

**Proof.** ($\Rightarrow$) $D$ is strongly connected by Proposition 2.2. Assume that there exists a transition $t\in T_D={}^{\bullet}D$ with $|{}^{\bullet}t\cap D|\geqslant 2$. Let $p\in|{}^{\bullet}t\cap D|$. Since $N$ is free choice, we have $p^{\bullet}=\{t\}$. It follows that ${}^{\bullet}(D\backslash\{p\})\subseteq {}^{\bullet}D\subseteq D^{\bullet}=(D\backslash\{p\})^{\bullet}$. Hence, $D\backslash\{p\}$ is a deadlock, what contradicts the minimality of $D$.

($\Leftarrow$) Assume $D$ is not minimal. Then there exists a minimal deadlock $D'\subset D$. By Proposition 2.2, $D'$ is strongly connected. Hence the subnet $N_{D'}=(P_{D'},T_{D'},F_{D'})$ generated by $D'\cup {}^{\bullet}D'$ is strongly connected. Moreover, since $D'\neq D$, we have $N_{D'}\subset N_D$. In consequence, there exists an arc $(x,y)\in F_D$, where $y\in P_{D'}\cup T_{D'}$, such that $(x,y)\notin F_{D'}$. $y$ cannot be a place, because otherwise $N_{D'}$ is not generated by $D'\cup {}^{\bullet}D'$. Hence, $y$ is a transition of $T_D$ and $|{}^{\bullet}y\cap D|\geqslant 2$. □

This theorem leads to an algorithm that constructs a minimal deadlock containing a given place. We need the following definition.

**Definition 6.5.** Let $N_1=(P_1,T_1,F_1)$ be a partial subnet of a net $N$. An elementary path $(x_1,\ldots,x_r)$, $r\geqslant 2$, of $N$ is a *handle* of $N_1$ iff $\{x_1,\ldots,x_r\}\cup(P_1\cap T_1)=\{x_1,x_r\}$.

The algorithm is very similar to the one proposed in [2] for the calculation of $T$-components.

**Algorithm 6.6.** To construct a minimal deadlock containing a given place.

*Input*: a strongly connected free choice net $N=(P,T,F)$ with a distinguished place $\tilde{p}$. This place $\tilde{p}$ is called the *seed* of the algorithm.

*Output*: a minimal deadlock of $N$ containing $\tilde{p}$.

We construct inductively a net $\tilde{N}=(\tilde{P}\subseteq P,\tilde{T}\subseteq T,\tilde{F}\subseteq F)$ such that $\tilde{P}$ will turn out to be a minimal deadlock of $N$. In the following the dot notation $^{\bullet}$ for pre- and post-sets always refers to the net $N$.

*Step 1*: $\tilde{P}:=\{\tilde{p}\}$, $\tilde{T}:=\emptyset$, $\tilde{F}:=\emptyset$ and $\tilde{N}:=(\tilde{P},\tilde{T},\tilde{F})$.

*Step 2*: Repeat the following exhaustively: If there is $p\in\tilde{P}$ and $t\in {}^{\bullet}p$ such that $(t,p)\notin\tilde{F}$, then choose a handle $H=(x_0,x_1,\ldots,x_{m-1},x_m)$ of $\tilde{N}$ with $x_{m-1}=t$ and $x_m=p$ (note that $m\geqslant 1$ and the equality can occur). Then put:

$\tilde{P}:=\tilde{P}\cup\{\text{places of } H\}$
$\tilde{T}:=\tilde{T}\cup\{\text{transitions of } H\}$
$\tilde{F}:=\tilde{F}\cup\{\text{arcs of } H\}$
$\tilde{N}:=(\tilde{P},\tilde{T},\tilde{F})$.

Let us collect now five simple properties of the construction. The first three hold at every stage of the algorithm.

(1) $\tilde{N}$ is a partial subnet of $N$.

(2) $\tilde{N}$ is strongly connected in terms of $\tilde{F}$.

At the very beginning, $\tilde{N}$ is trivially strongly connected and adding handles to it does not destroy the strong connectedness.

(3) Every transition in $\tilde{T}$ has exactly one incoming $\tilde{F}$ arc.

It has at least one because $\tilde{N}$ is strongly connected and $\tilde{N}$ can not contain isolated transitions. It has at most one, because this is trivally true at the very beginning, and the addition of the particular handles considered in the algorithm does not destroy this property: the new transitions added by the handle have at most one incoming arc, because handles are by definition elementary paths. And, since the last node of the handles added to $\tilde{N}$ is always a place, no transition already present in $\tilde{N}$ can find properly increased its number of incoming arcs by the addition of the new handles.

(4) At the end of the algorithm (which clearly terminates, due to the finiteness of $N$), if $p \in \tilde{P}$ then all the incoming arcs of $p$ in $F$ are also in $\tilde{F}$ (and therefore, ${}^{\bullet}p \subseteq \tilde{T}$).

The reason is that there always exists, at each stage of the algorithm, at least one handle satisfying the requirements: this derives easily from the strong connectedness of $N$.

(5) At the end of the algorithm $\tilde{N}$ is a subnet of $N$ (and $\tilde{N}$ is generated by $\tilde{P} \cap \tilde{T}$).

Assume the contrary. Then there exists an arc $f \in F$ between two nodes of $\tilde{N}$ such that $f \notin \tilde{F}$. Two possibilities have to be considered: $f$ leads from a transition to a place or from a place to a transition. The first is easily discarded because it contradicts property 4. Consider the second: if $f$ leads from a place to a transition, since $\tilde{N}$ is strongly connected it has to be the case that $|p^{\bullet}| > 1$ and $|{}^{\bullet}t| > 1$ (recall that the dot notation always refers to $N$). Then $N$ is not free choice.

**Theorem 6.7.** *Let $N = (P, T, F)$ be a strongly connected free choice net, $\tilde{p} \in P$ a place of $N$ and $\tilde{N} = (\tilde{P}, \tilde{T}, \tilde{F})$ a net constructed using Algorithm 6.6 with $\tilde{p}$ as seed. Then $\tilde{P}$ is a minimal deadlock of $N$.*

**Proof.** Since $\tilde{T} \subseteq \tilde{P}^{\bullet}$ by construction, and $\tilde{T} = {}^{\bullet}\tilde{P}$ (property 4), it follows that ${}^{\bullet}\tilde{P} \subseteq \tilde{P}^{\bullet}$. Hence, $\tilde{P}$ is a deadlock of $N$. Moreover, $\tilde{P}$ is a strongly connected deadlock because $\tilde{N}$ is the subnet generated by $\tilde{P} \cup \tilde{T} = \tilde{P} \cup {}^{\bullet}\tilde{P}$ (property 5) and $\tilde{N}$ is strongly connected (property 2). Finally, every transition $t \in \tilde{T}$ satisfies $|{}^{\bullet}t \cap \tilde{P}| = 1$ (property 3). By Theorem 6.4, $\tilde{P}$ is a minimal deadlock of $N$. □

Let us consider now the relationship between minimal and strongly connected deadlocks in free choice nets. We need the following lemma.

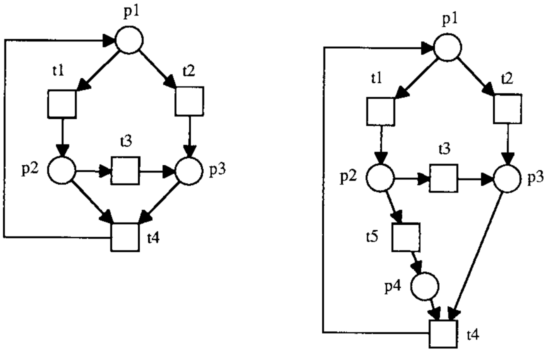


Fig. 9. Illustration of Theorem 6.9.

**Lemma 6.8.** *Let $N=(P, T, F)$ be a net, $P' \subseteq P$ and $N_{P'}$ the subnet generated by $P' \cup {}^{\bullet}P'$. Then $D' \subseteq P'$ is a deadlock of $N$ iff it is a deadlock of $N_{P'}$.*

**Proof.** Easy, using ${}^{\bullet}D' \subseteq {}^{\bullet}P'$ and the definition of deadlock. □

**Theorem 6.9.** *Let $N=(P, T, F)$ be a free choice net and $D \subseteq P$ a strongly connected deadlock of $N$. Then $D$ is the union of a set of minimal deadlocks of $N$.*

**Proof.** Let $N_D=(D, T_D, F_D)$ be the subnet of $N$ generated by $D \cup {}^{\bullet}D$. $N_D$ is strongly connected by definition and is obviously also free choice. Using Algorithm 6.6, given $p \in D$ it is possible to construct a minimal deadlock $D_p$ of $N_D$ containing $p$. We prove that $D_p$ is also a minimal deadlock of $N$. Using Lemma 6.8 with $D=P'$, we obtain that $D_p$ is a deadlock of $N$. Assume $D_p$ is not minimal in $N$. Then it contains a minimal deadlock $D'$. But, again by Lemma 6.8, $D'$ is also a deadlock of $N_D$, and since $D' \subseteq D_P$ this contradicts the hypothesis that $D_p$ was a minimal deadlock of $N_D$. Therefore $D_p$ is a minimal deadlock of $N$. Since $D=\bigcup_{p \in D} D_p$, we are done. □

Fig. 9 illustrates this result. Consider the net of Fig. 9(a), which is not free choice. $D=\{p_1, p_2, p_3\}$ is a strongly connected deadlock. Nevertheless, $D$ cannot be covered by minimal deadlocks, because the only minimal deadlock is $\{p_1, p_2\}$. Now add a transition $t_5$ and a place $p_4$ to make the net free choice (Fig. 9(b)). $D'=\{p_1, p_2, p_3, p_4\}$ is again a strongly connected deadlock, but now $D'$ can be covered by the minimal deadlocks $\{p_1, p_2, p_4\}$ and $\{p_1, p_2, p_3\}$.

**Theorem 6.10.** *In a bounded free choice system, every minimal deadlock is a trap iff every strongly connected deadlock is a trap.*

**Proof** ($\Rightarrow$) By Proposition 2.3, if minimal deadlocks are traps, their unions are traps as well. But by Theorem 6.9 the set of these unions contains the set of strongly connected deadlocks.

($\Leftarrow$) Use proposition 2.4. □

**Corollary 6.11.** *A bounded free choice system is live iff every strongly connected deadlock is a marked trap.*

**Proof.** Use Corolllary 6.3 and Theorem 6.10. □

Fig. 10 shows that Corollary 6.11 is false for non-bounded free choice systems. The set $\{p_1, p_2, p_4\}$ is a strongly connected deadlock but not a trap.

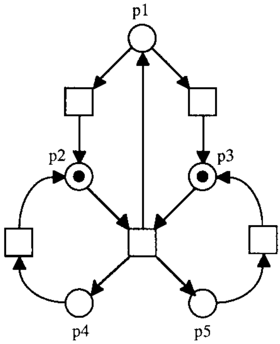


Fig. 10. $\{p_1, p_2, p_4\}$ is a strongly connected deadlock but not a trap. Nevertheless, the system is live.

**Theorem 6.12.** *Let $(N, M_0)$ be a bounded free choice system. It can be decided in polynomial time if $(N, M_0)$ is live.*

**Proof.** Use Theorem 5.2 and Corollary 6.11. □

The net system of Fig. 7 is bounded free choice. It was shown in Section 5 that it contains a strongly connected deadlock that is not a trap. By Corollary 6.11, the system is not live.

## Conclusions

In [11] a new technique for the computation of deadlocks and traps was proposed. We have shown here that the technique calculates exactly the unions of strongly connected deadlocks or traps of the net. We have also given a new proof of correctness that solves some small technical problems of the old proof. Our characterization of the computable deadlocks leads to a polynomial time algorithm that decides if every deadlock of a given system is a marked trap. Since the algorithm requires to solve sets of linear inequalities, its polynomiality derives from the polynomiality of linear programming. It is well known that the polynomial algorithms for linear programming behave in practice worse than the simplex. The average complexity of our algorithm using simplex will have to be empirically estimated on a certain selection of examples. Using some new results concerning the properties of minimal deadlocks in free choice nets, we have shown that our algorithm decides the liveness of bounded free choice systems. This result solves a conjecture raised by Jones et al. in [10].

## Appendix: Basic notations

A *net* is a triple $N=(P, T, F)$ with $P\cap T=\emptyset$ and $F\subseteq(P\times T)\cup(T\times P)$. $P$ is the set of *places*, $T$ the set of *transitions* and $F\subseteq(P\times T)\cup(T\times P)$ is the *flow relation.* The same symbol $F$ is used for the flow relation and its characteristic function on $(P\times T)\cup(T\times P)$.

The elements of $P\cup T$ are called *nodes.* $N$ is *pure* iff $\forall x, y\in P\cup T$: $(x, y)\in F\Rightarrow (y, x)\notin F$.

The *pre-set* of $x\in P\cup T$ is ${}^{\bullet}x=\{y\in P\cup T\,|\,(y, x)\in F\}$. The *post-set* of $x\in P\cup T$ is $x^{\bullet}=\{y\in P\cup T\,|\,(x, y)\in F\}$. The pre- and post-sets of a set of nodes are the union of the pre- and post-sets of its elements. A node $x$ is *isolated* iff ${}^{\bullet}x=\emptyset=x^{\bullet}$.

A function $M: P\to\mathbb{N}$ is called a *marking.* A *net system*, or *system* for short, is a pair $(N, M_0)$ where $N$ is a net and $M_0$ a marking of $N$ called initial marking.

A transition $t\in T$ is *enabled at M* iff $\forall p\in{}^{\bullet}t$: $M(p)\geqslant 0$. If $t$ is enabled at $M$, then $t$ may *fire* or *occur*, yielding a new marking $M'$ (denoted $M[t\rangle M'$), where $M'(p)= M(p)+F(t, p)-F(p, t)$.

A sequence of transitions, $\sigma=t_1t_2\dots t_r$ is an *occurrence sequence* of $(N, M_0)$ iff there exists a sequence $M_0t_1M_1t_2M_2\dots t_rM_r$ such that $\forall i$, $1\leqslant i\leqslant r$: $M_{i-1}[t_i\rangle M_i$. The marking $M_r$ is said to be *reachable* from $M_0$ by the occurrence of $\sigma$: (denoted $M[\sigma\rangle M_r$). $[M_0\rangle$ is the set of all markings reachable from $M_0$.

A system $(N, M_0)$ is *bounded* iff $\exists k\in\mathbb{N}\forall p\in P\forall M\in[M_0\rangle$: $M(p)\leqslant k$. $(N, M_0)$ is live iff $\forall t\in T\forall M\in[M_0\rangle\exists M'\in[M\rangle$: $M'$ enables $t$. $(N, M_0)$ is *deadlock-free* iff $\forall M\in [M_0\rangle$: $\exists t\in T$ enabled at $M$.

A net $N=(P, T, F)$ is a *P-graph* iff $\forall t\in T$: $|{}^{\bullet}t|=|t^{\bullet}|=1$. $N$ is a *T-graph* iff $\forall p\in P$: $|{}^{\bullet}p|=|p^{\bullet}|=1$. $N$ is *free choice* iff $\forall p\in P$ such that $|p^{\bullet}|>1$: ${}^{\bullet}(p^{\bullet})=\{p\}$. $N$ is *asymmetric choice* iff $\forall t\in T$: $|\{p\in{}^{\bullet}t\,|\,|p^{\bullet}|>1\}|\leqslant 1$.

$N=(P', T', F')$ is a *subnet* of $N=(P, T, F)$ (denoted $N'\subseteq N$) iff $P'\subseteq P$, $T'\subseteq T$ and $F'=F\cap((P'\times T')\cup(T'\times P'))$. $N'$ is said to be *generated* by $P\cup T'$. $N'$ is a *partial* subnet of $N$ (denoted $N'\leqslant N$) iff $P'\subseteq P$, $T'\subseteq T$ and $F'\subseteq F\cap((P'\times T')\cup(T'\times P'))$.

A *path* of $N$ is a nonempty sequence $(x_1, x_2, \dots, x_r)$ of elements of $X=P\cup T$ such that $\forall i$, $1\leqslant i\leqslant r-1$: $(x_i, x_{i+1})\in F$. A path is *elementary* iff all $x_i$ are distinct, except possibly $x_1$ and $x_r$. A *circuit* of $N$ is a path $(x_1, \dots, x_r)$ such that $x_1=x_r$. A circuit is *elementary* iff it is elementary as a path.

Let $N=(P, T, F)$ be a net with $P=\{p_1,\ldots,p_n\}$, $T=\{t_1,\ldots,t_m\}$. The matrix $C=\|c_{ij}\|$ $(1\leq i\leq n, 1\leq j\leq m)$ where $c_{ij}=F(t_j, p_i)-F(p_i, t_j)$ is the *incidence matrix* of $N$. A nonnegative integer vector $X$ is a *P-semiflow* of $N$ iff $X\neq 0$ and $X^T\cdot C=0^T$. The set $\|X\|=\{p\in P\,|\,X(p)>0\}$ is the *support* of $X$. A $P$-semiflow $X$ is *minimal* iff there is no $P$-semiflow $Y\neq X$ such that $\|Y\|\subseteq|X\|$.

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