Repeating Structural Results

Class Definitions

$$
\begin{aligned}
L &= DSPACE(\log n) \\
NL &= NSPACE(\log n) \\
P &= \bigcup_{c=1}^{\infty} DTIME(n^c) \\
NP &= \bigcup_{c=1}^{\infty} NTIME(n^c) \\
PSPACE &= \bigcup_{c=1}^{\infty} DSPACE(n^c) \\
NPSPACE &= \bigcup_{c=1}^{\infty} NSPACE(n^c) \\
EXP &= \bigcup_{c=1}^{\infty} DTIME(2^{n^c}) \\
NEXP &= \bigcup_{c=1}^{\infty} NTIME(2^{n^c})
\end{aligned}
$$

Relationships
(By Definition)

Determinism vs. Nondeterminism	Exponentially Higher Bound
$L \subseteq NL$	$L \subseteq PSPACE$
$P \subseteq NP$	$NL \subseteq NPSPACE$
$PSPACE \subseteq NSPACE$	$P \subseteq EXP$
$EXP \subseteq NEXP$	$NP \subseteq NEXP$

Three Structural Theorems

$NTIME(f(n)) \subseteq DSPACE(f(n))$

$NSPACE(f(n)) \subseteq DTIME(c^{\log n + f(n)})$

$NSPACE(f(n)) \subseteq DSPACE(f^2(n))$
$f(n) \geq \log n$

f time/space constructible

NTIME vs. DSPACE: Proof I

$NTIME(f(n)) \subseteq DSPACE(f(n))$

Let M be an NTM running in time $f(n)$.

In each step, M can make a single nondeterministic decision.

However, M can only chose out of c_M continuations in a step.

Thus, $\overline{M}$ enumerates all possible choices, taking space $c_M\, f(n)$.

This string is then used by $\overline{M}$ as a lookup-table
whenever M is taking a nondet. choice.

NTIME vs. DSPACE: Proof II

Thus, $\overline{M}$ enumerates all possible choices, taking space $c_M\, f(n)$.

This string is then used by $\overline{M}$ as a lookup-table
whenever M is taking a nondet. choice.

For each enumerated choice-string, $\overline{M}$ simulates M.

If M aceepts in one of these simulations, $\overline{M}$ accepts, too.

Otherwise, $\overline{M}$ rejects.

$\overline{M}$ requires $c_M\, f(n) + f(n)$ space, i.e. $\overline{M} \in DSPACE(f(n))$. ●

NSPACE vs. DTIME: Proof I

$NSPACE(f(n)) \subseteq DTIME(c^{\log n + f(n)})$

Let M be an NTM running in space $f(n)$.
A configuration of M has the following parts:

- the state $k \in K_M$ of M
- the cursor position $1 \leq i \leq n+1$ of M
- the contents $< s_1, \ldots, s_l >$ of the tapes of M : $s_i \in \Sigma^{f(n)}$

Thus, there are $|K_M|(n+1)|\Sigma|^{lf(n)}$ different configs.
Using C_M we find at most $C_M^{\log n + f(n)}$ configs.

NSPACE vs. DTIME: Proof II

Using C_M we find at most $C_M^{\log n + f(n)}$ configs.

Now we define $G_x^M = < V, E >$ with $V = \{$configs. of $M\}$ and $< u, v > \in E$ iff there is a direct transition from u to v on input x.

Define $s \in V$ to be the initial config of M and $t \in V$ to be the final config of M (normalization).

$< G_x^M, s, t >$ is a $REACH$ instance with $C_M^{\log n + f(n)}$ nodes.
$< G_x^M, s, t > \in REACH$ iff $M(x) = 1$

NSPACE vs. DTIME: Proof III

$< G_x^M, s, t >$ is a $REACH$ instance with $C_M^{\log n + f(n)}$ nodes.
$< G_x^M, s, t > \in REACH$ iff $M(x) = 1$

$REACH \in P$. Thus we can decide $< G_x^M, s, t > \in REACH$ in $DTIME((C_M^{\log n + f(n)})^k)$ for some constant k.

$DTIME((C_M^{\log n + f(n)})^k) = DTIME(c^{\log n + f(n)})$ ●

NSPACE vs. DTIME
A Note on the Proof

$< G_x^M, s, t >$ is a $REACH$ instance with $C_M^{\log n + f(n)}$ nodes.
$< G_x^M, s, t > \in REACH$ iff $M(x) = 1$

The method of representing a space-bounded computation by a graph G_x^M is called the "Reachability-Method".

Effectively, this is a generic reduction!
$REACH$ is $NL-hard$.

We will come back to this issue – and clarify in detail!

NSPACE vs. DSPACE: Proof I

$NSPACE(f(n)) \subseteq DSPACE(f^2(n))$
$f(n) \geq \log n$

$< G_x^M, s, t >$ is a $REACH$ instance with $C_M^{\log n + f(n)}$ nodes.
$< G_x^M, s, t > \in REACH$ iff $M(x) = 1$

since $f(n) \geq \log n$

$< G_x^M, s, t >$ is a $REACH$ instance with $C^{f(n)}$ nodes.
$< G_x^M, s, t > \in REACH$ iff $M(x) = 1$

NSPACE vs. DSPACE: Proof II

$< G_x^M, s, t >$ is a $REACH$ instance with $C^{f(n)}$ nodes.
$< G_x^M, s, t > \in REACH$ iff $M(x) = 1$

We cannot compute the graph – it is exponential!
So how to access it?

We can compute the configuartions s and t.

Having two nodes u and v, we check $< u, v > \in E$ by simulating M on u with input string x.

NSPACE vs. DSPACE: Proof III

```
PATH(G,i,j,d)
    if <i,j> ∈ E then return true;
    if d = 0 then return false;
    for(z = 1; z < |V|; ++z)
        if PATH(G,i,z,d-1) and PATH(G,z,j,d-1) then
            return true;
    return false;
```

$PATH(G,i,j,d)$ is true iff $\exists$ a path from i to j of length $\leq 2^d$

$PATH(G,s,t,\lceil \log |V| \rceil)$ iff $<G,s,t> \in REACH$

NSPACE vs. DSPACE: Proof IV

```
PATH(G,i,j,d)
    if <i,j> ∈ E then return true;
    if d = 0 then return false;
    for(z = 1; z < |V|; ++z)
        if PATH(G,i,z,d-1) and PATH(G,z,j,d-1) then
            return true;
    return false;
```

Recursive depth of at most d

Each "stack-frame" has size $3\log|V|$

$PATH(G,s,t,\lceil \log |V| \rceil)$ requires $3\log^2|V|$ space

NSPACE vs. DSPACE: Proof V

$<G_x^M,s,t>$ is a $REACH$ instance with $C_M^{f(n)}$ nodes.

$<G_x^M,s,t> \in REACH$ iff $M(x)=1$

$PATH(G,s,t,\lceil \log |V| \rceil)$ iff $<G,s,t> \in REACH$

$PATH(G,s,t,\lceil \log |V| \rceil)$ requires $3\log^2|V|$ space

Taken together : $M(x)=1$ can be decided in

$DSPACE(3\log^2(C_M^{f(n)})) = DSPACE(f^2(n))$ ●

Three Structural Theorems: Consequences

Theorem	Consequences
$NTIME(f(n)) \subseteq DSPACE(f(n))$	$NP \subseteq PSPACE$
$NSPACE(f(n)) \subseteq DTIME(c^{\log n + f(n)})$	$NL \subseteq P$ $NPSPACE \subseteq EXP$
$NSPACE(f(n)) \subseteq DSPACE(f^2(n))$ $f(n) \geq \log n$	$NPSPACE \subseteq PSPACE$

f time/space constructible

Relationships

Determinism vs. Nondeterminism	Exponentially Higher Bound
$L \subseteq NL$	$L \subseteq PSPACE$
$P \subseteq NP$	$NL \subseteq NPSPACE$
$PSPACE \subseteq NSPACE$	$P \subseteq EXP$
$EXP \subseteq NEXP$	$NP \subseteq NEXP$

Relationships

Determinism vs. Nondeterminism	The three Theorems
$L \subseteq NL$	$NL \subseteq P$
$P \subseteq NP$	$NP \subseteq PSPACE$
$PSPACE \subseteq NSPACE$	$NPSPACE \subseteq PSPACE$
$EXP \subseteq NEXP$	$NPSPACE \subseteq EXP$

Relationships

$L \subseteq NL$

$NL \subseteq P$

$P \subseteq NP$

$NP \subseteq PSPACE$

$PSPACE \subseteq NSPACE$

$NPSPACE \subseteq PSPACE$

$EXP \subseteq NEXP$

$NPSPACE \subseteq EXP$

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP \subseteq NEXP$$

Diagonalization

Time Hierarchy Theorem

$$DTIME(f(n)) \subset DTIME(f(n)^c)$$
$$f(n) \geq n, \text{ time constr., for some } c > 1$$

Bounded Simulation

Let $B_f^{DTIME} = \{< M, x > | M(x) = 1 \text{ within } DTIME(f(|x|))\}$

$B_f^{DTIME} \in DTIME(s[f](n))$

$f(n) \geq n, \text{ time constr.}$

Time Hierarchy: Proof I

Let $B_f^{DTIME} = \{< M, x > | M(x) = 1 \text{ within } DTIME(f(|x|))\}$

$B_f^{DTIME} \in DTIME(s[f](n))$ (Bounded Simulation)

Set $D_f^{DTIME} = \{M \mid < M, M > \notin B_f^{DTIME}\}$

Let N be an arbitrary Machine in $DTIME(f(n))$

$N \in L_N \Leftrightarrow < N, N > \in B_f^{DTIME}$

$N \in L_N \Leftrightarrow N \notin D_f^{DTIME}$

$D_f^{DTIME} \notin DTIME(f(n))$ $D_f^{DTIME} \in DTIME(s[f](2n+1))$

Time Hierarchy: Proof II

Let $B_f^{DTIME} = \{< M, x > | M(x) = 1 \text{ within } DTIME(f(|x|))\}$

$B_f^{DTIME} \in DTIME(s[f](n))$ (Bounded Simulation)

$D_f^{DTIME} \notin DTIME(f(n))$ $D_f^{DTIME} \in DTIME(s[f](2n+1))$

$$DTIME(f(n)) \subset DTIME(s[f](2n+1))$$

There are several bounded simulation results. It is important to us that $s[f](n)$ is bounded by a polynomial in f. E.g., $s[f](n) = f^3(n)$, for $f(n) \geq n$

$$DTIME(f(n)) \subset DTIME(f^3(2n+1))$$

Time Hierarchy Theorem

$$DTIME(f(n)) \subset DTIME(f(n)^c)$$
$$f(n) \geq n, \text{ time constr., for some } c > 1$$

.... constants do not matter, only the polynomial relationsship is important

Hierarchy Theorem: Reusing the Proof

$D_f^{RES} \notin RES(f(n))$ $\qquad$ $D_f^{RES} \in RES(s[f](2n+1))$

The last proof was generic – every deterministic bounded simulation can be substituted.

$$B_f^{DSPACE} \in DSPACE(f(n)\log f(n))$$

$$DSPACE(f(n)) \subset DSPACE(f(n)\log f(n))$$

Also possible to prove a hierarchy for NTIME, NSPACE….

Hierarchy Theorems: Relativization

$D_f^{RES} \notin RES(f(n))$ $\qquad$ $D_f^{RES} \in RES(s[f](2n+1))$

... holds with respect to any oracle A, i.e.,

$$DTIME^A(f(n)) \subset DTIME^A(f^c(n))$$

$$DSPACE^A(f(n)) \subset DSPACE^A(f(n)\log f(n))$$

Relationships

$L \subseteq PSPACE$

$NL \subseteq NPSPACE$

$P \subseteq EXP$

$NP \subseteq NEXP$

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP \subseteq NEXP$$

Relationships

$L \subset PSPACE$

$NL \subset NPSPACE$

$P \subset EXP$

$NP \subset NEXP$

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP \subseteq NEXP$$

Relationships

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP \subseteq NEXP$$

$NL \subset PSPACE$ $\qquad$ $P \subset EXP$ $\qquad$ $NP \subset NEXP$

Thus there must be proper inclusions. But nobody was able to find them…

Time Constructible

- computing f should be easy
- more precisely doable in DTIME(f)
- a machine should be able to measure the "time"
- "yard-stick" machines

Time Constructible Definition

Let f be a function $N \to N$ with

$f(n+1) \geq f(n)$

If there is a DTM M which ouputs $1^{f(n)}$ on input $x\ (|x| = n)$ and runs within $DTIME(n + f(n))$

then f is time constructible

Space Constructible Definition

Let f be a function $N \to N$ with

$f(n+1) \geq f(n)$

If there is a DTM M which ouputs $1^{f(n)}$ on input $x\ (|x| = n)$ and runs within $DSPACE(f(n))$

then f is space constructible

Time/Space Constructible Functions: Examples

$f(n) = c$ $\qquad$ $f(n) = n!$

$f(n) = \log(n)$ $\qquad$ $f(n) = \sqrt{n}$

$f(n) = n$

$\left.\begin{matrix} f(n) + g(n) \\ f(n)g(n) \\ f(n)^{g(n)} \end{matrix}\right\} f(n)$ and $g(n)$ proper

Important proper complexity functions

Time/Space Constructible What for?

Where did we need them?

Hierarchy Theorems
Theorem of Immerman

Gap-Theorem

An Anomaly: The Gap-Theorem

Let g be a recursive function $N \to N$ with $g(n+1) > g(n)$. Then there is a recursive function $f: N \to N$ with $DTIME(f(n)) = DTIME(g(f(n)))$.

Gap Theorem for DTIME

Same for DSPACE, NTIME, NSPACE…

Gap-Theorem Proof I

$g(n+1) > g(n)$

$G[0](x) = g(x)$

$G[i+1](x) = g(G[i](x))$

$P(n,k) \Leftrightarrow \forall 0 \le i \le n \quad \forall x \in \Sigma^n$

$M[i](x)$ in $DTIME(G[k])$ or

$M[i](x)$ notin $DTIME(G[k+1])$

$M[i]$ enumerates the DTMs.

Gap-Theorem Proof II

$g(n+1) > g(n)$

$G[0](x) = g(x)$

$G[i+1](x) = g(G[i](x))$

$P(n,k) \Leftrightarrow \forall 0 \le i \le n \quad \forall x \in \Sigma^n$

$M[i](x)$ in $DTIME(G[k])$ or

$M[i](x)$ notin $DTIME(G[k+1])$

$G[0]$ $G[1]$ $G[2]$ $G[3]$

$M[0](x)$ $M[1](y)$ | $M[1](x)$ | $P(1,2) = T$ | $M[0](y)$

Gap-Theorem Proof III

$P(n,k) \Leftrightarrow \forall 0 \le i \le n \quad \forall x \in \Sigma^n$

$M[i](x)$ in $DTIME(G[k])$ or

$M[i](x)$ notin $DTIME(G[k+1])$

$f(n) = G[k]$ where k min. with $P(n,k) = T$

$G[0]$ $G[1]$ $G[2]$ $G[3]$

$M[0](x)$ $M[1](y)$ | $M[1](x)$ | $P(1,2) = T$ | $M[0](y)$

Gap-Theorem Proof IV

$f(n) = G[k]$ where k min. with $P(n,k) = T$

$M[i]$ in $DTIME(g(f(n)))$

then $M[i](x)$ for $|x| \ge i$ in $DTIME(f(n))$ •

$G[0]$ $G[1]$ $G[2]$ $G[3]$

$M[0](x)$ $M[1](y)$ | $M[1](x)$ | $P(1,2) = T$ | $M[0](y)$